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ADDITIVELY TRANSTABLE CONIC SECTIONS WITH RESPECT TO FIXED COEFFICIENTS

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Abstract. The arithmetic mean has several important properties. One of them preserves the result of the arithmetic mean. That is, if one value increases and another decreases, the result of the arithmetic mean is the same. This property is called transfer stability, transtability for short. We can see its reach in several mathematical theories. The most common use is with aggregation functions. This article aims to show another use of this property, specifically in the geometry of conic sections. We have outlined how the transtability of a conic section works. The main idea was to find a common property for conic sections connected by transtability. We found that these conics have the same common intersection and the set of all centers forms a conic.
Keywords: Arithmetic Mean, Conic Sections, Geometric Properties.

1. Introduction

Transfer-stability (newly transtability) was introduced in the paper [11] to convert the arithmetic mean into the lattice theory [7, 8]. The main problem in this task arose with additivity, i.e., $F(x+y) = F(x) + F(y)$ for $x, y \in \mathbb{R}$. Additivity cannot be transferred to lattices, so it had to be replaced by another property

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of the arithmetic mean that can be used in lattice theory. This problem gave rise to transtability, more precisely, additive transtability, because we are adding and subtracting the same number. Its primary use was for aggregation functions [2, 3, 4, 6, 10, 12]. However, it turned out that the application of aggregation functions possessing this property extends to the area of multi-criteria decision-making [13].

A well-known use of transtability is in the theory of polynomials. We say that the polynomial $f(x)$ is transtable with the polynomial $g(x)$ if the polynomial $g(x)$ was created from the polynomial $f(x)$ by adding the constant $k \in \mathbb{R}$ to one coefficient, and by subtracting the same constant k from another coefficient. In such a case, the sum of the coefficients is preserved, and therefore, we can express a more straightforward definition that says that two polynomials are transtable if they have the same sum of coefficients. The class of transtable polynomials has several interesting properties. The first, already mentioned, property is that transtable polynomials form an equivalence class or congruence. In other words, transtability is equivalence and also congruence with respect to the addition and multiplication of polynomials. The second property is, for example, that the class $\mathbf{0}$ (the sum of the coefficients is equal to 0) contains all polynomials with a root equal to 1. The last property mentioned here is the existence of a polynomial with integer roots in each class.

In this paper, we focus on the use of additive transtability (transtability for short, we will not use any other than additive transtability) in geometry. Specifically, in the theory of plane conic sections [1], instead of polynomials, as mentioned in the previous paragraph. Similar to polynomials, we can increase and decrease the coefficients arbitrarily, but we ran into a problem. We have not found any common property that unites such conic sections. Therefore, we focused on the shift of only two preselected coefficients.

The goal of this paper is to introduce the notion of transtability in geometry, specifically in the theory of conic sections, and to show some interesting properties that these conic sections satisfy.

An interesting fact at the end. The term transtability was first used by Rufus Isaacs in 1949 in [9]. It was followed up a few years later by Sidney C. Dixon in [5] for the National Aeronautics and Space Administration (NASA). Rufus Isaacs wrote: "The term transtability refers to the loss of stable static equilibrium of a buckled panel when the speed of flow exceeds a certain critical value (transtability value)." It is apparent that "universal" transtability has nothing to do with our "mathematical" transtability.

2. Transtability

In the introduction, we described the concept of transtability as increasing or decreasing coefficients or maintaining the sum of coefficients. We also stated that transtability for conic sections did not lead to anything interesting, but we defined this term formally for completeness of this paper.

Definition 2.1. Two conic sections K and L are said to be *transtable* if the sum of their coefficients is the same, i.e., if the equations of conic sections K and L, are given in the forms

$$\begin{aligned} K: \quad & a_1x^2 + b_1y^2 + c_1xy + d_1x + e_1y + f_1 = 0, \\ L: \quad & a_2x^2 + b_2y^2 + c_2xy + d_2x + e_2y + f_2 = 0, \end{aligned}$$

respectively, then it holds $a_1 + b_1 + c_1 + d_1 + e_1 + f_1 = a_2 + b_2 + c_2 + d_2 + e_2 + f_2$.

We can also refer to the term in the Definition 2.1 as *complete transtability.* It means that transtability is not restricted in any way. It is straightforward that the entire set of transtable conic sections forms an equivalence class, which is determined by a real number representing the sum of coefficients. If we mark such a class with the symbol $[k]$, then we can say that the conic section K with the sum of coefficients equal to k belongs to this class, i.e., $K \in [k]$. In other words, each real number corresponds to one equivalence class. Therefore, the number of classes equals the number of real numbers. In Figure 2.1, we can see several conic sections from the class [1]. For illustration, we also state their equations

$$\begin{aligned} 1x^2 + 1y^2 - 1 &= 0, \\ 11x^2 - 9y^2 - 1 &= 0, \\ -9x^2 + 11y^2 - 1 &= 0, \\ 11x^2 + 1y^2 - 11 &= 0, \\ 1x^2 + 11y^2 - 11 &= 0, \\ -9x^2 + 1y^2 + 9 &= 0, \\ 1x^2 - 9y^2 + 9 &= 0, \end{aligned}$$

where we see the shift of the two different coefficients by 10 based on the first equation.

However, according to the Definition 2.1, we can increase or decrease any two coefficients. Therefore, Figure 2.2 shows a more illustrative structure of the class [1] with the following conic sections equations

$$\begin{aligned} 11x^2 + 1y^2 - 10xy - 1 &= 0, \\ -9x^2 + 1y^2 + 10xy - 1 &= 0, \\ 11x^2 + 1y^2 - 10x - 1 &= 0, \\ -9x^2 + 1y^2 + 10x - 1 &= 0, \\ 11x^2 + 1y^2 - 10y - 1 &= 0, \\ -9x^2 + 1y^2 + 10y - 1 &= 0, \\ 1x^2 + 11y^2 - 10xy - 1 &= 0, \\ 1x^2 - 9y^2 + 10xy - 1 &= 0, \end{aligned}$$

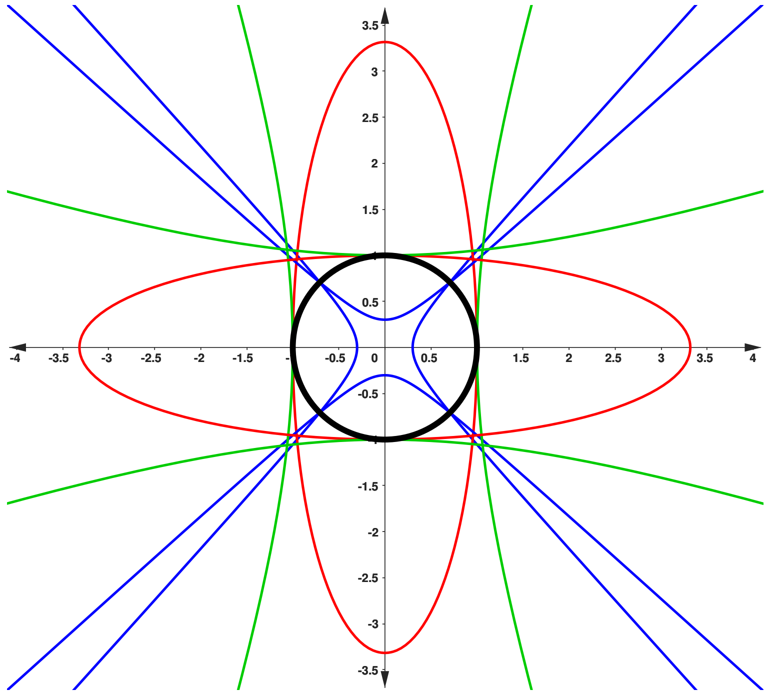

Fig. 2.1: Transtable class [1] – addition to/substraction from non-zero coefficients

$$
\begin{aligned}
1x^2 + 11y^2 - 10x - 1 &= 0,\\
1x^2 - 9y^2 + 10x - 1 &= 0,\\
1x^2 + 11y^2 - 10y - 1 &= 0,\\
1x^2 - 9y^2 + 10y - 1 &= 0,\\
1x^2 + 1y^2 + 10xy - 10x - 1 &= 0,\\
1x^2 + 1y^2 - 10xy + 10x - 1 &= 0,\\
1x^2 + 1y^2 + 10xy - 10y - 1 &= 0,\\
1x^2 + 1y^2 - 10xy + 10y - 1 &= 0,\\
1x^2 + 1y^2 + 10xy - 11 &= 0,\\
1x^2 + 1y^2 - 10xy + 9 &= 0,\\
1x^2 + 1y^2 + 10x - 11 &= 0,\\
1x^2 + 1y^2 - 10x + 9 &= 0,\\
1x^2 + 1y^2 + 10y - 11 &= 0,\\
1x^2 + 1y^2 - 10y + 9 &= 0.
\end{aligned}
$$

This process led us to the fact that we could not find any reasonable property that would unite all these conic sections. Therefore, we weakened the transtabil-

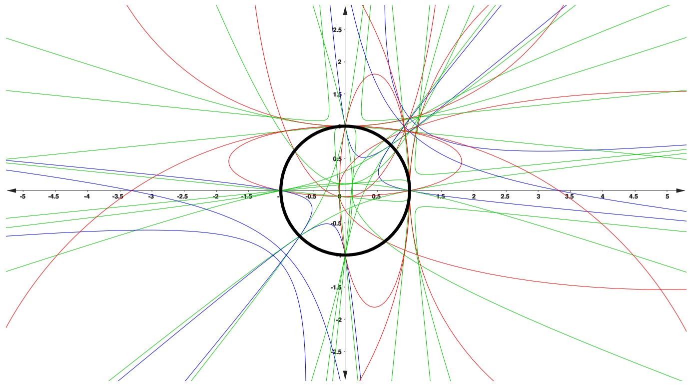

FIG. 2.2: Transtable clas [1] – addition to/substraction from any coefficient

ity and selected only two fixed coefficients. We get the following definition when choosing the first two coefficients, i.e., x^2 and y^2.

Definition 2.2. A conic section K in the form $ax^2+by^2+cxy+dx+ey+f=0$ is said to be *transtable with respect to* x^2 *and* y^2 with a conic section L if there exists $k \in \mathbb{R}$ such that the conic section L has the form $(a+k)x^2+(b-k)y^2+cxy+dx+ey+f=0$.

Based on the above mentioned definition, we are limited by the previously chosen coefficients a and b. In this case, we shall call *partial transtability*. In the sense of Definition 2.1, it means that we consider only the sums $a_1+b_1=a_2+b_2$. Therefore, complete transtability implies partial transtability, but the reverse is not valid.

In Figure 2.3 we see a transtable class with respect to x^2 and y^2 determined by a partial sum equal to 2, based on the equation of the circle $x^2+y^2-1=0$. Additional conic sections are obtained by gradually decreasing and increasing the coefficients of x^2 and y^2 by the value of 1.

Note that Definition 2.2 induces another 14 partial transtabilities. For example, with respect to x^2 and xy, or with respect to x and y, or with respect to y and *const.*, etc. In total, there is one complete transtability and 15 partial transtabilities for conic sections.

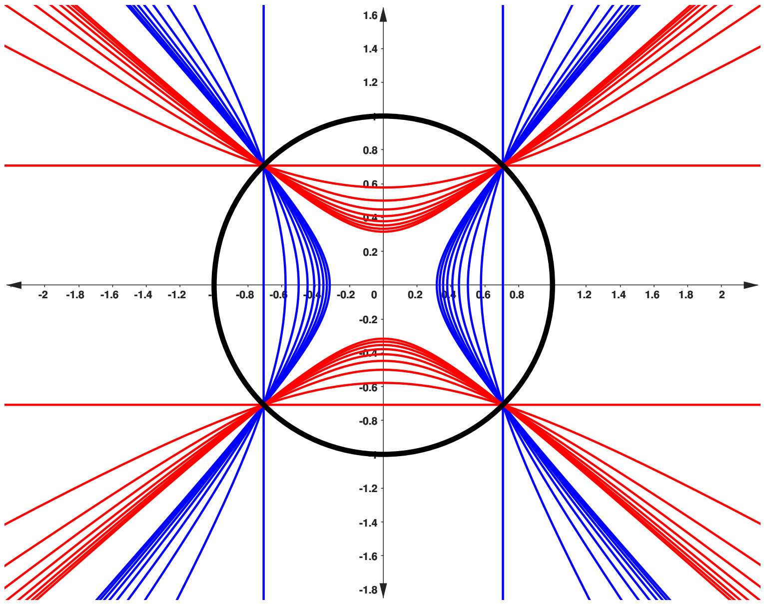

FIG. 2.3: Transtable class [2] with respect to x^2 and y^2

3. Intersection of partially transtable conic sections

We will now focus on the intersections of partially transtable conic sections. As previously stated, completely transtable conic sections have no common intersection. In other words, the intersection of all completely transtable conic sections is an empty set. On the other hand, for partially transtable conic sections, the intersection may not be an empty set, which depends on the choice of coefficients and the given conic section. We will show that the number of intersection points can be at most 4 (in some cases, at most 2) and at least none.

Theorem 3.1. *Each pair of partially transtable conic sections with respect to the given coefficients has at most* 4 *common points and may have no common point.*

Proof. The proof is divided into 15 parts, for each partial transtability. In all the following partial transtabilities, we will consider the conic section equation

$$ax^2 + by^2 + cxy + dx + ey + f = 0. \tag{3.1}$$

□

3.1. Transtability with respect to x^2 and y^2

We will focus on the coefficients a and b in Equation (3.1). Consider two arbitrary transtable conic sections K and L with respect to x^2 and y^2. Without loss of generality, assume that their equations are given in the forms, respectively

$$\begin{aligned} K: &\quad ax^2 + by^2 + cxy + dx + ey + f = 0, \\ L: &\quad (a+k)x^2 + (b-k)y^2 + cxy + dx + ey + f = 0. \end{aligned}$$

We get the relations $y = \pm x$ from the sum of these two equations. Substituting the first relation into the first conic section equation, we obtain the two intersection points

$$P_{1,2}^{x^2,y^2} = [x, x],$$

where the coordinate x is calculated from the quadratic equation $(a+b+c)\,x^2 + (d+e)\,x + f = 0$. The next two intersection points are

$$P_{3,4}^{x^2,y^2} = [x, -x],$$

where the coordinate x is calculated from the quadratic equation $(a+b-c)\,x^2 + (d-e)\,x + f = 0$.

Example 3.1. Let

$$K_1 : 2x^2 + 3y^2 - 3xy - 6x + 9y - 9 = 0$$

be an arbitrary conic section (in this case, an ellipse). Then, it is evident that conic sections

$$\begin{aligned} K_2: &\quad 1x^2 + 4y^2 - 3xy - 6x + 9y - 9 = 0, \\ K_3: &\quad 0x^2 + 5y^2 - 3xy - 6x + 9y - 9 = 0, \\ K_4: &\quad -1x^2 + 6y^2 - 3xy - 6x + 9y - 9 = 0, \\ K_5: &\quad 3x^2 + 2y^2 - 3xy - 6x + 9y - 9 = 0, \\ K_6: &\quad 4x^2 + 1y^2 - 3xy - 6x + 9y - 9 = 0, \\ K_7: &\quad 5x^2 + 0y^2 - 3xy - 6x + 9y - 9 = 0, \end{aligned}$$

are transtable with the conic section K_1 with respect to x^2 and y^2 (see Figure 3.1).

All these conic sections intersect at exactly four points. We get the first two points from the solution of the quadratic equation $(2+3-3)x^2 + (-6+9)x - 9 = 0$, where the roots are $x_1 = -3$ and $x_2 = \frac{3}{2}$. So the intersections have the form $P_1^{x^2,y^2} = [-3, 3]$, and $P_2^{x^2,y^2} = \left[\frac{3}{2}, \frac{3}{2}\right]$.

We calculate the next two points from the quadratic equation $(2+3+3)x^2+(-6-9)x-9 = 0$, where the solutions are $x_1 = \frac{3\left(5-\sqrt{57}\right)}{16}$ and $x_2 = \frac{3\left(5+\sqrt{57}\right)}{16}$. Thus, the intersections have the form $P_3^{x^2,y^2} = \left[\frac{3\left(5-\sqrt{57}\right)}{16}, -\frac{3\left(5-\sqrt{57}\right)}{16}\right]$, and $P_4^{x^2,y^2} = \left[\frac{3\left(5+\sqrt{57}\right)}{16}, -\frac{3\left(5+\sqrt{57}\right)}{16}\right]$.

Example 3.2. Let

$$K_1 : 2x^2 - y^2 - 8xy + 2x + 9y - 10 = 0$$

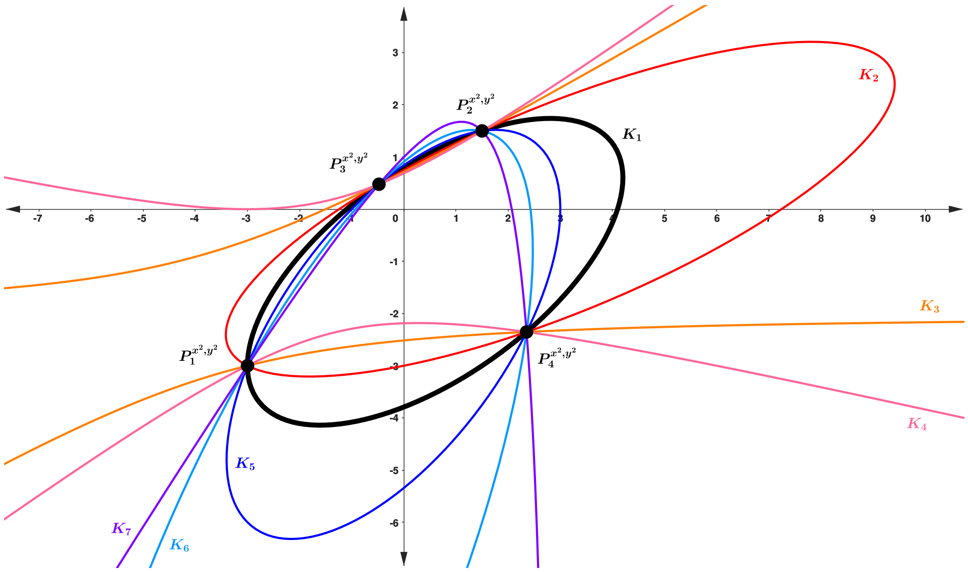

FIG. 3.1: Illustration of Example 3.1

be an arbitrary conic section (in this case, a hyperbola). Then, it is straightforward to see that the conic sections

$$\begin{aligned}
K_2:&\quad 1x^2+0y^2-8xy+2x+9y-10=0,\\
K_3:&\quad 0x^2+1y^2-8xy+2x+9y-10=0,\\
K_4:&\quad -1x^2+2y^2-8xy+2x+9y-10=0,\\
K_5:&\quad 3x^2-2y^2-8xy+2x+9y-10=0,\\
K_6:&\quad 4x^2-3y^2-8xy+2x+9y-10=0,\\
K_7:&\quad 5x^2-4y^2-8xy+2x+9y-10=0,
\end{aligned}$$

are transtable with respect to x^2 and y^2 with the conic section K_1 (see Figure 3.2).

All these conic sections intersect at exactly two points. We do not get any points from solving the quadratic equation $(2-1-8)x^2+(2+9)x-10=0$ because the roots are complex. Thus, the intersections $P_{1,2}^{x^2,y^2}$ do not exist.

We calculate the next two intersection points from the quadratic equation $(2-1+8)x^2+(2-9)x-10=0$, where the solutions are $x_1=\frac{7-\sqrt{409}}{18}$ and $x_2=\frac{7+\sqrt{409}}{18}$. Thus, the intersection points are $P_3^{x^2,y^2}=\left[\frac{7-\sqrt{409}}{18},-\frac{7-\sqrt{409}}{18}\right]$, and $P_4^{x^2,y^2}=\left[\frac{7+\sqrt{409}}{18},-\frac{7+\sqrt{409}}{18}\right]$.

Example 3.3. Let

$$K_1: 2x^2-3y^2+0xy-1x+2y-3=0$$

be an arbitrary conic section (in this case, a hyperbola). Then it is obvious that the conic sections

$$K_2:\quad 1x^2-2y^2+0xy-1x+2y-3=0,$$

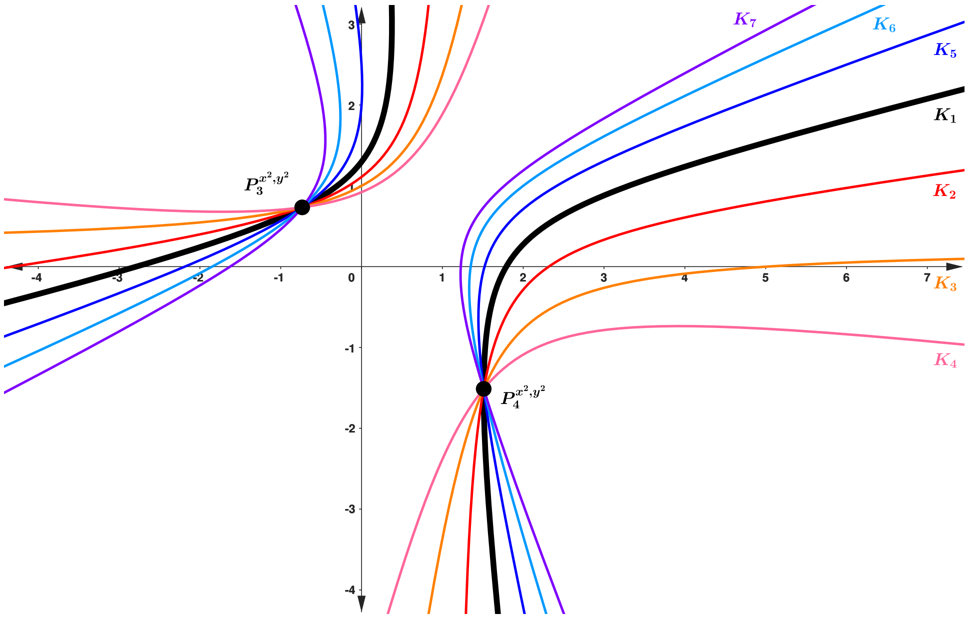

FIG. 3.2: Illustration of Example 3.2

$$
\begin{aligned}
K_3: &\quad 0x^2 - 1y^2 + 0xy - 1x + 2y - 3 = 0,\\
K_4: &\quad -1x^2 - 0y^2 + 0xy - 1x + 2y - 3 = 0,\\
K_5: &\quad 3x^2 - 4y^2 + 0xy - 1x + 2y - 3 = 0,\\
K_6: &\quad 4x^2 - 5y^2 + 0xy - 1x + 2y - 3 = 0,\\
K_7: &\quad 5x^2 - 6y^2 + 0xy - 1x + 2y - 3 = 0,
\end{aligned}
$$

are transtable with respect to x^2 and y^2 with the conic section K_1 (see Figure 3.3).

All these conic sections have no intersection point because the quadratic equations $(2-3+0)x^2 + (-1+2)x - 3 = 0$ and $(2-3-0)x^2 + (-1-2)x - 3 = 0$ have complex roots, i.e., this class of partially transtable conic sections has an empty intersection.

3.2. Transtability with respect to x^2 and xy

We will focus on the coefficients a and c in Equation (3.1). Consider two arbitrary transtable conic sections K and L with respect to x^2 and xy. Without loss of generality, assume that they are, respectively, in the form

$$
\begin{aligned}
K: &\quad ax^2 + by^2 + cxy + dx + ey + f = 0,\\
L: &\quad (a+k)x^2 + by^2 + (c-k)xy + dx + ey + f = 0.
\end{aligned}
$$

We get the relation $x(x-y) = 0$ from the sum of these two equations. Substituting $y = x$ or $x = 0$ into one of the conic section equations, we obtain the coordinates

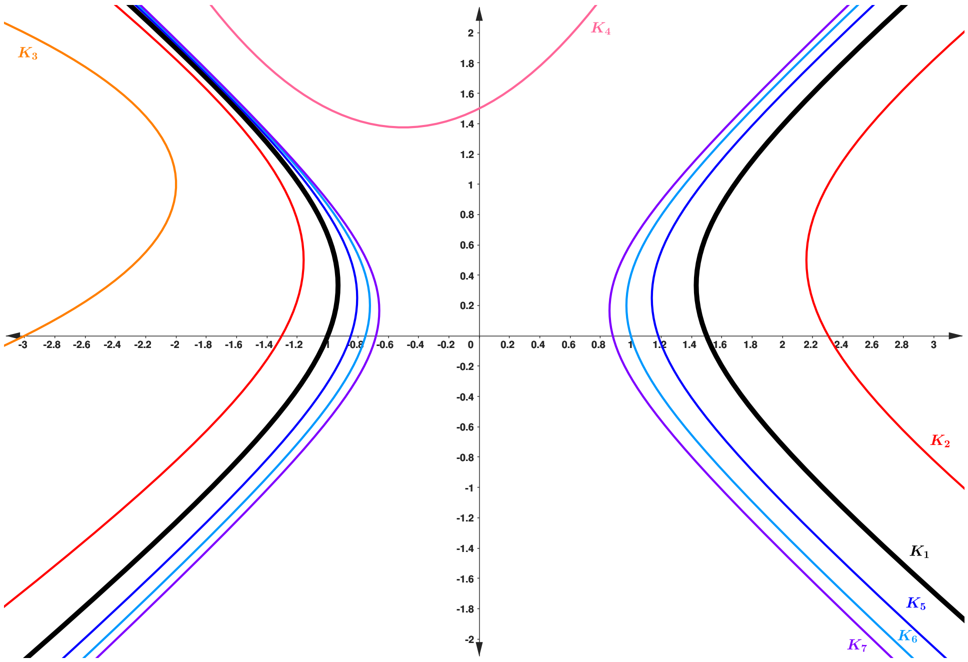

FIG. 3.3: Illustration of Example 3.3

of the intersection points. First two intersection points $P_{1,2}^{x^2,xy}$ correspond to the intersection points $P_{1,2}^{x^2,y^2}$ in Section 3.1.. The next two intersection points are

$$P_{3,4}^{x^2,xy} = [0, y]\,,$$

where the y-coordinate is calculated from the quadratic equation $by^2 + ey + f = 0$.

Analogy

The case of transtability with respect to y^2 and xy is analogous. The first two intersection points are the same, i.e., $P_{1,2}^{y^2,xy} = P_{1,2}^{x^2,xy}$ and the other two intersection points are

$$P_{3,4}^{y^2,xy} = [x, 0]\,,$$

where the x-coordinate is calculated from the quadratic equation $ax^2 + dx + f = 0$.

3.3. Transtability with respect to x^2 and x

We will focus on the coefficients a and d in Equation (3.1). Consider two arbitrary transtable conic sections K and L with respect to x^2 and x. Without loss of generality, assume that they are given, respectively, by the equations

$$K : ax^2 + by^2 + cxy + dx + ey + f = 0,$$
$$L : (a + k)x^2 + by^2 + cxy + (d - k)x + ey + f = 0.$$

From the sum of these two equations, we get $x(x - 1) = 0$. Substituting $x = 1$ in one of the conic section equations, we obtain the coordinates of the intersection points

$$P_{1,2}^{x^2,x} = [1, y],$$

where the y-coordinate is calculated from the quadratic equation $by^2 + (c + e)\, y + a + d + f = 0$. The next two intersection points $P_{3,4}^{x^2,x}$ correspond to the intersection points $P_{3,4}^{x^2,xy}$ in Section 3.2.. We get them after substituting $x = 0$ in the equation of a conic section.

Analogy

The case of trasntability with respect to y^2 a y is analogous. The first two intersection points are

$$P_{1,2}^{y^2,y} = [x, 1],$$

where the x-coordinate is calculated from the quadratic equation $ax^2 + (c + d)\, x + b + e + f = 0$. The other two intersection points correspond to the intersection points in the analogous part of Section 3.2., i.e., $P_{3,4}^{y^2,y} = P_{3,4}^{y^2,xy}$.

3.4. Transtability with respect to x^2 and y

Let us focus on the coefficients a and e in Equation (3.1). Consider two arbitrary transtable conic sections K and L with respect to x^2 and y. Without loss of generality, assume that their equations are given, respectively, in the form

$$K : \quad ax^2 + by^2 + cxy + dx + ey + f = 0,$$
$$L : \quad (a + k)x^2 + by^2 + cxy + dx + (e - k)y + f = 0.$$

The sum of both equations yields $y = x^2$. Substituting this relation into one of the conic section equations, we obtain the coordinates of the intersection points

$$P_{1,2,3,4}^{x^2,y} = \left[x, x^2\right],$$

where the coordinate x is calculated from the fourth order equation $bx^4 + cx^3 + (a + e)\, x^2 + dx + f = 0$.

Analogy

The case of transtability with respect to y^2 a x is analogous. All four intersections, if they exist, are

$$P_{1,2,3,4}^{y^2,x} = \left[y^2, y\right],$$

where the coordinate y is calculated from the fourth order equation $ay^4 + cy^3 + (b + d)\,y^2 + ey + f = 0$.

3.5. Transtability with respect to x^2 and *const*.

Let us focus on the coefficients a and f in Equation (3.1). Consider two arbitrary transtable conic sections K and L with respect to x^2 and *const*. Without loss of generality, assume that their equations are given, respectively, in the form

$$\begin{aligned} K: &\quad ax^2 + by^2 + cxy + dx + ey + f = 0,\\ L: &\quad (a+k)x^2 + by^2 + cxy + dx + ey + (f-k) = 0. \end{aligned}$$

We get $(x + 1)(x - 1) = 0$ from the sum of these equations. Substituting $x = \pm 1$ into one of the conic section equations, we obtain the coordinates of the intersection points. The first two intersection points $P_{1,2}^{x^2,const}$ correspond to the intersection points $P_{1,2}^{x^2,x}$ from Section 3.3., i.e., $P_{1,2}^{x^2,const} = P_{1,2}^{x^2,x}$. The next two intersection points are

$$P_{3,4}^{x^2,const} = [-1, y],$$

where the coordinate y is calculated from the quadratic equation $by^2 + (-c + e)\,y + a - d + f = 0$.

Analogy

The case of transtability with respect to y^2 and *const*. is analogous. The first two intersection points $P_{1,2}^{y^2,const}$ are the same as intersection points $P_{1,2}^{y^2,y}$ from the analogic part of Setion 3.3.. The next two intersection points are

$$P_{3,4}^{y^2,const} = [x, -1],$$

where the coordinate x is calculated from the quadratic equation $ax^2 + (-c + d)\,x + b - e + f = 0$.

3.6. Transtability with respect to xy and x

Let us focus on the coefficients c and d in Equation (3.1). Consider two arbitrary transtable conic sections K and L with respect to xy and x. Without loss of generality, assume that their equations are given, respectively, in the form

$$K: \quad ax^2 + by^2 + cxy + dx + ey + f = 0,$$
$$L: \quad ax^2 + by^2 + (c+k)xy + (d-k)x + ey + f = 0.$$

From the sum of both equations, we get $x(y-1) = 0$. Substituting $y = 1$ or $x = 0$ into one of the conic section equations, we obtain the coordinates of the already known intersection points. The first two intersection points $P_{1,2}^{xy,x}$ correspond to the intersection points $P_{1,2}^{y^2,y}$ in the analogous part of Section 3.3.. The other two intersection points $P_{3,4}^{xy,x}$ correspond to the intersection points $P_{3,4}^{x^2,xy}$ in Section 3.2..

Analogy

The case of transtability with respect to xy and y is completely analogous. The first two intersection points $P_{1,2}^{xy,y}$ are the same as the intersection points $P_{1,2}^{x^2,x}$ from Section 3.3.. The other two intersection points $P_{3,4}^{xy,y}$ are the same as the intersection points $P_{3,4}^{y^2,xy}$ from the analogous part in Section 3.2..

3.7. Transtability with respect to xy and $const$.

Let us consider the coefficients c and f in Equation (3.1). Consider two arbitrary transtable conic sections K and L with respect to xy and $const$. Without loss of generality, assume that they are given, respectively, by the equations

$$K: \quad ax^2 + by^2 + cxy + dx + ey + f = 0,$$
$$L: \quad ax^2 + by^2 + (c+k)xy + dx + ey + (f-k) = 0.$$

From the sum of both equations, we get $xy = 1$. Substituting $y = \frac{1}{x}$ into one of the conic section equations, we obtain the coordinates of all four intersection points

$$P_{1,2,3,4}^{xy,const} = \left[x, \frac{1}{x}\right],$$

where the coordinate x is calculated from the fourth order equation $ax^4 + dx^3 + (c+f)\,x^2 + ex + b = 0$.

3.8. Transtability with respect to x and y

Let us focus on the coefficients d and e in Equation (3.1). Consider two arbitrary transtable conic sections K and L with respect to x and y. Without loss of generality, assume that they are given, respectively, by the equations

$$\begin{aligned} K: &\quad ax^2 + by^2 + cxy + dx + ey + f = 0, \\ L: &\quad ax^2 + by^2 + cxy + (d+k)x + (e-k)y + f = 0. \end{aligned}$$

From the sum of both equations, we get $y = x$. Substituting $y = x$ into one of the conic section equations, we obtain the coordinates of only two already known intersection points $P_{1,2}^{x,y}$, which correspond to the intersection points $P_{1,2}^{x^2,y^2}$ from Section 3.1.. No other intersection points exist.

3.9. Transtability with respect to x and *const*

We focus on the coefficients d and f in Equation (3.1). Consider two arbitrary transtable conic sections K and L with respect to x and *const*. Without loss of generality, assume that they are given, respectively, by the equations

$$\begin{aligned} K: &\quad ax^2 + by^2 + cxy + dx + ey + f = 0, \\ L: &\quad ax^2 + by^2 + cxy + (d+k)x + ey + (f-k) = 0. \end{aligned}$$

From the sum of both equations, we get the equality $x = 1$. By substituting this equality into one of the conic section equations, we obtain the coordinates of the intersection points. Similar to the previous case, there are only two intersection points $P_{1,2}^{x,const}$ that correspond to the intersection points $P_{1,2}^{x^2,x}$ from Section 3.3..

Analogy

The case of trasntability with respect to y and *const*. is analogous. There exist only two intersection points $P_{1,2}^{y,const}$ corresponding to $P_{1,2}^{y^2,y}$ in analogous part of Section 3.3.. $\square$

To conclude this section, let us note that the class $[0]$ of transtable conic sections is distinguished by the point $[1,1]$ because this point represents the entire class. In other words, the conic section $K \in [0]$ if and only if the point $[1,1] \in K$. Moreover, it is the only point common to all partial transtabilities, i.e., the only point that can become an intersection point in every partial transtability mentioned above. The only condition for this is the already mentioned zero sum of coefficients. The question arises whether the point $[1,1]$ is the only one that possesses such a property, whether it is a representation of a given class or a common intersection point in all partial transtabilities. The answer to the second question is trivial and is YES. No other point can become a common intersection point in all the abovementioned cases. The answer to the first question is also positive, which is proved by the following Lemma.

Lemma 3.1. *Let $A = [x, y]$ be an arbitrary point such that $A \neq [1, 1]$. Then there exist two conic sections $K \in [k]$ and $L \in [l]$ such that $k \neq l$ and $A \in K, L$.*

Proof. Consider a fixed point $A = [x, y]$ and two conic sections K and L given by the equations

$$\begin{aligned} K: &\quad a_1x^2 + b_1y^2 + c_1xy + d_1x + e_1y + f_1 = 0, \\ L: &\quad a_2x^2 + b_2y^2 + c_2xy + d_2x + e_2y + f_2 = 0. \end{aligned}$$

We divide the proof into two parts. In the first part, suppose that $x \neq y$. Then, for the conic section K, we choose the coefficients as follows

$$a_1 = b_1 = c_1 = e_1 = 0, d_1 = 1, f_1 = -x,$$

where $A \in K$ holds. Moreover, the sum of coefficients is equal to $a_1 + b_1 + c_1 + d_1 + e_1 + f_1 = 1 - x$. For the conic section L, we choose the coefficients as follows

$$a_2 = b_2 = c_2 = d_2 = 0, e_2 = 1, f_2 = -y,$$

where also $A \in L$ holds. Moreover, the sum of all coefficients is equal to $a_2 + b_2 + c_2 + d_2 + e_2 + f_2 = 1 - y$. From the assumption $x \neq y$ it follows $1 - x \neq 1 - y$. Thus, we found two conic sections with different sums of coefficients that pass through the point A.

In the second part, suppose that $x = y$, $x \neq 1$. The outcome is similar to the first part. Let us choose the coefficients as follows

$$\begin{aligned} a_1 = b_1 = c_1 = 0,\ d_1 = 1,\ e_1 = -1,\ f_1 = 0 \\ a_2 = b_2 = c_2 = 0,\ d_2 = 1,\ e_2 = 1,\ f_2 = -2x. \end{aligned}$$

Then the conic sections K and L have the form

$$\begin{aligned} K: &\quad x - y = 0, \\ L: &\quad x + y - 2x = 0, \end{aligned}$$

where the condition $A \in K, L$ is fullfilled. Furthermore, the sum of coefficients of the conic sections K and L, respectively, equals 0 and $1 + 1 - 2x$. Thanks to the assumption $x \neq 1$, we get two different sums of coefficients. Thus Lemma is proved. □

4. Centres of partially transtable conic sections

In the previous section, we found that all conic sections have common intersection points, at most four different points, or else they have an empty intersection. In the second main topic of this paper, we will focus on the centres of partially transtable conic sections. Similarly to the previous section, we will consider the cases of all 15 partial transtabilities, where we will show the specific form of the set of all centres for the given case.

Let us remind that the centre of a conic section $ax^2+by^2+cxy+dx+ey+f=0$ has the coordinates

$$\left[\frac{ce-2bd}{4ab-c^2}; \frac{cd-2ae}{4ab-c^2}\right],$$

if it exists (i.e., a parabola does not have a centre). If we calculate the centre of all partially transtable conic sections, we get a set of centres. Below, we show that the set of centres is also a conic section (regular or singular, i.e., line, point, or an empty set).

Theorem 4.1. *The set of all centres of partially transtable conic sections with respect to the given coefficients is a conic section.*

Proof. The proof is divided into 15 parts, for each partial transtability. In all the following partial transtabilities, we will consider the conic section equation (3.1). □

4.1. Transtability with respect to x^2 and y^2

Let C_{x^2,y^2} be a set of all centres of transtable conic sections with respect to x^2 and y^2. Then C_{x^2,y^2} is a conic section given by the equation

$$-cx^2 - cy^2 - 2(b+a)xy - ex - dy = 0.$$

Example 4.1. In Exmaple 3.1, for the given conic section

$$K_1 : 2x^2 + 3y^2 - 3xy - 6x + 9y - 9 = 0$$

the set of centres C_{x^2,y^2} of all conic sections transtable with K_1 with respect to x^2 a y^2 is a hyperbola (see Figure 4.1) given by the equation

$$C_{x^2,y^2} : 3x^2 + 3y^2 - 10xy - 9x + 6y = 0.$$

Example 4.2. Similarly, for the conic section

$$K_1 : 2x^2 - y^2 - 8xy + 2x + 9y - 10 = 0$$

from Example 3.2 we get the set of centres C_{x^2,y^2} in the form

$$C_{x^2,y^2} : 8x^2 - 8y^2 - 2xy - 9x - 2y = 0$$

which is en ellipse (see Figure 4.2).

Example 4.3. Here, we do not refer to Example 3.3 since we would get a hyperbola as a set of centres. Insdead, let us consider a special case of set of cetners. Suppose

$$K_1 : x^2 - y^2 - 2 = 0$$

is a hyperbola. Then its centre is point $[0,0]$, which is also a centre of all its transtable conic sections with respect to x^2 a y^2 (See Figure 4.3).

Finally, let us note that for this partial transtability, the set of centres C_{x^2,y^2} cannot be empty. Even in the simplest examples, the same situation as above will occur. We get the centre at coordinate origin.

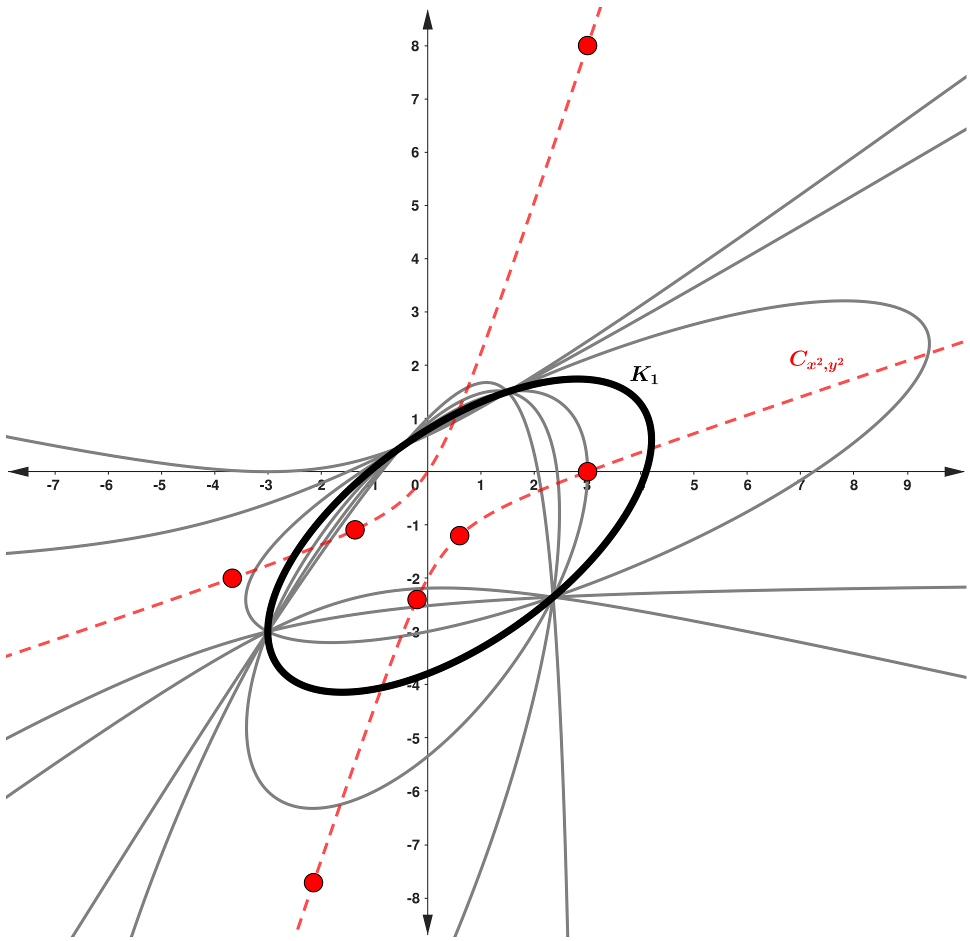

FIG. 4.1: Illustration of Example 4.1

4.2. Transtability with respect to x^2 and xy

Let $C_{x^2,xy}$ be a set of all centres of transtable conic sections with respect to x^2 and xy. Then $C_{x^2,xy}$ is a conic section given by the equation

$$2(a+c)x^2 - 2by^2 + 4bxy + (2e+d)x - ey = 0.$$

Analogy

Analogically, the set $C_{y^2,xy}$ of all centres of transtable conic sections with respect to y^2 and xy forms a conic section given by the equation

$$-2ax^2 + 2(b+c)y^2 + 4axy - dx + (2d+e)y = 0.$$

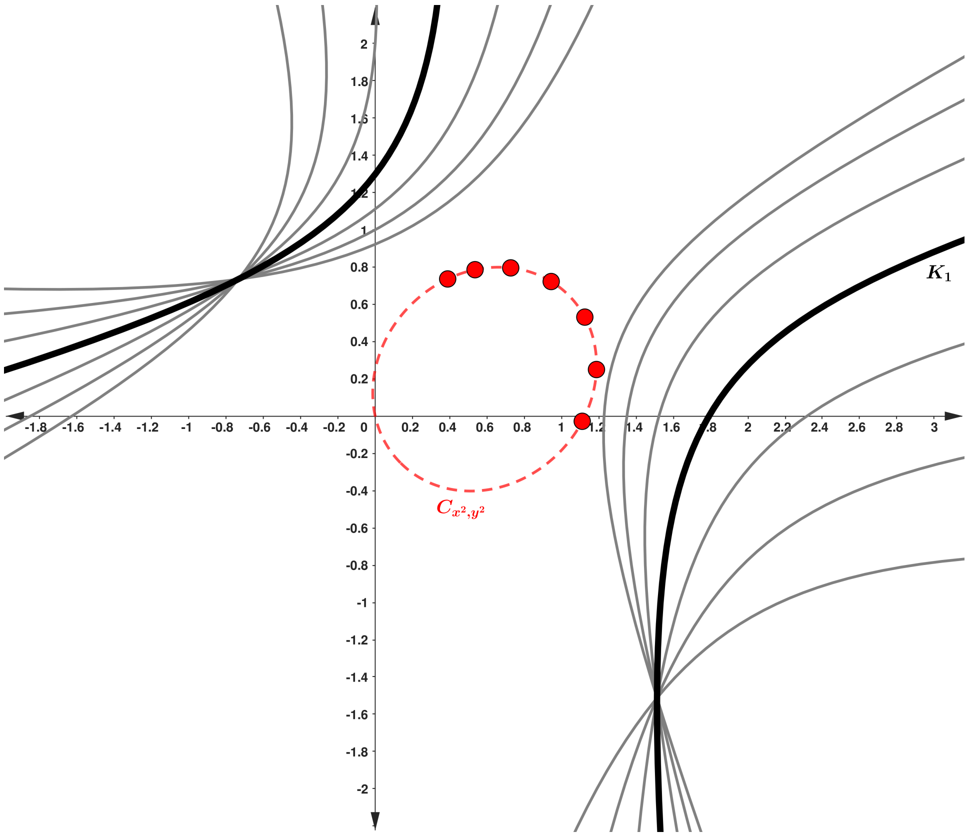

FIG. 4.2: Illustration of Example 4.2

4.3. Transtability with respect to x^2 and x

Let $C_{x^2,x}$ be a set of all centres of transtable conic sections with respect to x^2 and x. Then $C_{x^2,x}$ is a line given by the equation

$$cx + 2by + e = 0.$$

Analogy

Analogically, the set $C_{y^2,y}$ of all centres of transtable conic sections with respect to y^2 and y forms a line given by the equation

$$cy + 2ax + d = 0.$$

4.4. Transtability with respect to x^2 and y

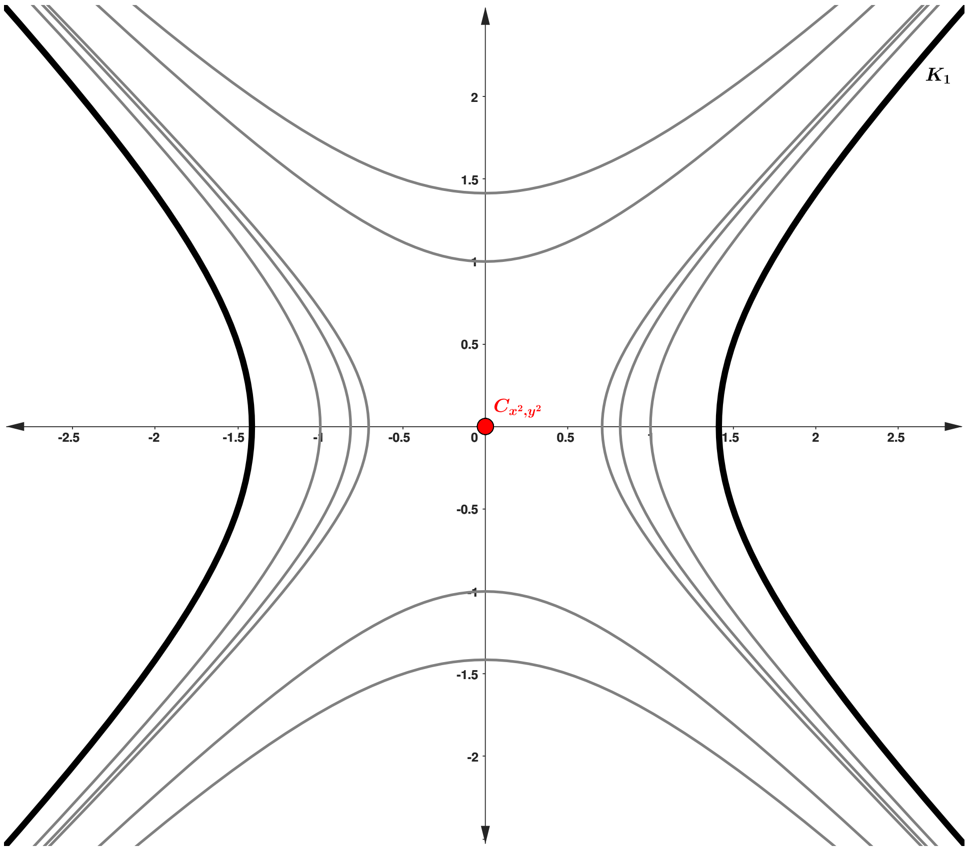

FIG. 4.3: Illustration off Example 4.3

Let $C_{x^2,y}$ be a set of all centres of transtable conic sections with respect to x^2 and y. Then $C_{x^2,y}$ is a conic section given by the equation

$$2cx^2 + 4bxy + 2(a+e)x + cy + d = 0.$$

Analogy

Analogically, the set $C_{y^2,x}$ of all centres of transtable conic sections with respect to y^2 and x forms a conic section given by the equation

$$2cy^2 + 4axy + 2(b+d)y + cx + e = 0.$$

4.5. Transtability with respect to x^2 and *const*.

Let $C_{x^2,const}$ be a set of all centres of transtable conic section with respect to x^2 and $const$. Then $C_{x^2,const}$ is a line that is the same as line $C_{y^2,y}$ (see Section 4.3.).

Analogy

Analogically, the set $C_{y^2,const}$ is a line that is the same as line $C_{y^2,y}$ (see Analogy in Section 4.3.).

4.6. Transtability with respect to xy and x

Let $C_{xy,x}$ be a set of all centres of transtable conic sections with respect to xy and x. Then $C_{xy,x}$ is a conic section given by the equation

$$-2ax^2 + 2by^2 - (d + c)xy + (-2b + e)y - e = 0.$$

Analogy

Analogically, the set $C_{xy,y}$ of all centres of transtable conic sections with respect to xy and y form a conic section given by the equation

$$2ax^2 + (-2b)y^2 + (-2a + d)x + (e + c)y - d = 0.$$

This case is the most interesting one among all partial transtabilities. The reason is that this is the only partial transtability where the set of centres can be a circle with centre at the coordinate origin and radius 1 (see Figure 4.4). In other cases, the set of centres can be a circle with any diameter but never with the centre at the coordinate origin. This exceptionality belongs only to this partial transtability.

4.7. Transtability with respect to x and y

Let $C_{x,y}$ be a set of all centres of transtable conic sections with respect to x and y. Then $C_{x,y}$ is a line in the form

$$(2a + c)x + (2b + c)y + d + e = 0.$$

4.8. Transtability with respect to xy and $const$.

Let $C_{xy,const}$ be a set of all centres of transtable conic sections with respect to xy and $const$. Then $C_{xy,const}$ is a conic section given by the equation

$$2ax^2 - 2by^2 + dx - ey = 0.$$

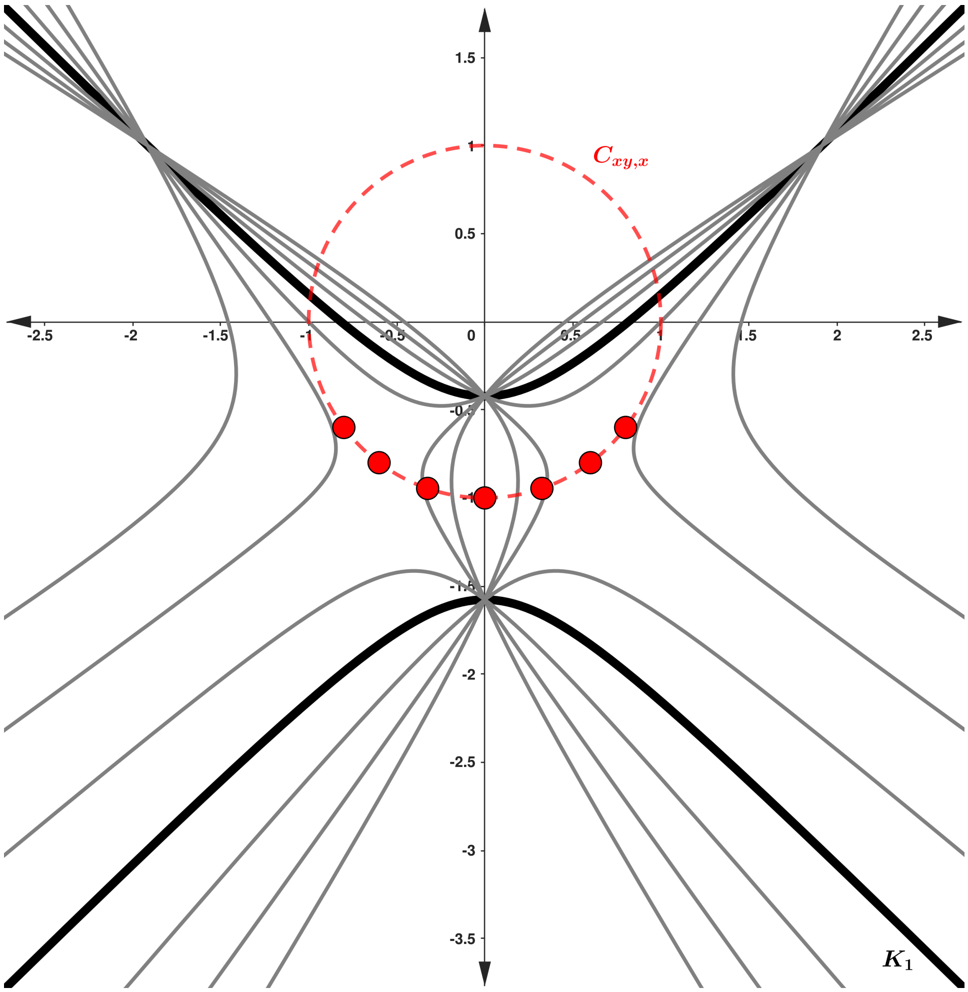

FIG. 4.4: Set of centres as a circle with a centre at the coordinate origin

4.9. Transtability with respect to x and *const*.

Let $C_{x,const}$ be a set of centres of transtable conic sections with respect to x and *const*. Then $C_{x,const}$ is a line, same as the line $C_{x^2,x}$ (see Section 4.3.).

Analogy

Analogically, the set $C_{y,const}$ is a line, same as the line $C_{y^2,y}$ (see Analogy in Section 4.3.). □

5. Conclusion

This paper aimed to introduce the concept of transtability outside the domain of aggregation functions. We focused on geometry, specifically on the theory of conic sections. In Section 2., we defined the already mentioned term of transtability. We pointed out that there are two options to look at this term, either as a whole or in parts, where the second-mentioned was more important for us since complete transtability did not give any interesting results.

In the following two sections, we have focused on two interesting properties we discovered using partial transtability. The first is the intersection of conic sections connected by partial transtability. In such a case, intersection occurs at most 4 points. What is important, however, is that there is some intersection because there was no such intersection in the case of complete transtability. As part of this section, we also showed how to find these intersections for individual cases. The second task was finding centres of conic sections in the same partially transtable class. We have shown that this set is not random but forms a conic section (regular or singular, i.e., a line, a point). Similar to the previous section, we have also shown all the equations for the sets of centres in individual cases.

This research has a broad spectrum of possible continuation, as we would like to show in which areas and theories of mathematics it is appropriate to use the concept of transtability. Whether the family of conic sections has any specific properties from a differential geometry point of view ([14]). In addition, the entire paper discussed only the so-called additive transtability, i.e., that we added and subtracted the same number to the coefficients of the conic section. In other words, the sum of the coefficients was the same. However, it turned out that similar properties appear even in the case of so-called multiplicative transtability, i.e., we multiply and divide the conic section coefficients by the same number. In other words, the product of the coefficients is the same.

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