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Fuzzy multiset regular languages and their basic characterizations

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ABSTRACT

The paper provides a survey of several ways how to describe fuzzy multiset regular languages, i.e., languages generated by fuzzy multiset regular grammars. These languages can also be characterized by means of fuzzy multiset finite automata (both in general and in reduced forms), fuzzy multiset regular expressions, and as fuzzy multiset languages which can be expressed in a semilinear form. Moreover, it is pointed out that a prevailing number of already published papers concerning fuzzy multiset finite automata is based on a wrong definition. It is also shown that the name 'deterministic fuzzy multiset finite automaton' is often used incorrectly for automata deserving adjective pseudodeterministic.

1. Introduction

Multisets (sometimes called bags, see, e.g., [1,2]) generalize sets in the respect that allow multiplied occurrence of their elements. This concept appears in various parts of mathematics, physics, philosophy, logic, linguistics, computer science ([3]), etc. We can mention, for example, the standard example of prime factorization which is naturally being expressed as a multiset of prime factors. As another example, we mention the multiset of threatened species where biologists are interested not only in a list of these species but also in numbers of their surviving members.

In Computer science, multisets appear repeatedly and with various motivations (see, e.g., [4,5] in visual languages processing, [6,7] with computational models inspired from biochemistry, [8,9] with computational models inspired from cell biology, etc.).

Collections of multisets are called multiset languages. In [10], a formal grammar approach was used to describe Chomsky-like hierarchy of multiset languages and it was supplemented with the corresponding multiset automata in [11]. In a multiset finite automaton, if a symbol x should be read, any x presented in the input multiset can be processed. Whilst finite automata (working over strings of symbols) process their inputs in the order given by positions of the symbols in an input string, multiset finite automata do not have any limitation of this kind. The feature not to consider position of a processed symbol in the input of an automaton can be found in systems enabling commutativity of input symbols. The commutativity appears for example in commutative grammars of [12] or in jumping grammars (see [13]) and jumping automata (see [14]). The basic difference between jumping automata and multiset automata lies in the form of their input. The former automata have input strings, the latter ones have input multisets. Clearly, a string of commuting symbols corresponds (via Parikh mapping) to the respective multiset. Hence, almost all results known for jumping automata can be transferred to multiset automata and vice versa. As an exception from this transfer, we can mention, for example, the operation of concatenation of string languages which does not relate to any operation of multiset languages.

Fuzzy approach to multiset languages was initiated by Wang et al. in [15] where fuzzy multiset grammars and fuzzy multiset finite automata were introduced (with the real unit interval $[0,1]$ as the set of truth values) and some closure properties of fuzzy multiset languages were studied. This paper was followed by [16] and [17] which dealt with deterministic variants of the automata.

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Minimization of fuzzy multiset finite automata was studied in papers like [17–19] — unfortunately, these papers are based on incorrectly proved propositions claiming equal computational power of nondeterministic and deterministic fuzzy multiset finite automata. Even their definitions of deterministic fuzzy multiset finite automata are not precise. More detailed notes to these weak points are in Section 3 of this paper.

The purpose of the presented paper is twofold. First, it describes fundamental characterizations of fuzzy multiset regular languages, which can be used in their further research. Second, it describes frequently used incorrect definitions of deterministic and nondeterministic (fuzzy) multiset finite automata, which settled in prevailed number of the related papers in last years and which led to some erroneous proofs.

The paper is organized as follows. Section 2 presents basic notions of multisets, operations on multisets, multiset finite automata, and multiset languages. Section 3 introduces fuzzy multiset regular languages as multiset languages generated by fuzzy multiset regular grammars and describes several kinds of fuzzy multiset finite automata. Notes concerning erroneous definitions of either nondeterministic and deterministic fuzzy multiset finite automata are added here. In Section 4, fuzzy multiset regular expressions are shown as another formalism enabling to describe fuzzy multiset regular languages. Section 5 characterizes fuzzy multiset regular languages as fuzzy multiset languages which can be expressed in semilinear form. Section 6 deals with two reduced forms of fuzzy multiset finite automata, the first with crisp transition relation and the second with crisp set of final states. The last section summarizes the results of this paper and presents a sketch of further research directions. A short note concerning invalidity of a transformation of Myhill-Nerode theorem to (fuzzy) multiset regular languages is mentioned as well.

2. Preliminaries

We assume certain familiarity of the reader with fundamentals of formal languages and automata theory (see, e.g., [20,21]). Therefore, we skip their classical notions and start with multisets, multiset grammars and multiset finite automata. For description of these basic notions, we use approach of [16].

We denote by $\mathbf{N}$ the set of all natural numbers including 0. If Σ is a finite nonempty set of symbols we call it an *alphabet*. The cardinality of any alphabet Σ is denoted by card(Σ).

For any alphabet Σ, a mapping $\sigma : \Sigma \to \mathbf{N}$ is called a finite *multiset*. Obviously, each set $U \subseteq \Sigma$ is a multiset σ_U such that $\sigma_U(x) = 1$ iff $x \in U$. We use notation of [22] and [23]. Thus, we denote the set of all multisets over Σ by $\Sigma^{\oplus}$. $\Sigma^{\oplus}$ is a commutative monoid with operation of *addition* $\oplus$ and neutral element $\mathbf{0}_\Sigma$, defined as follows:

$(\alpha \oplus \beta)(x) = \alpha(x) + \beta(x)$ for all $x \in \Sigma$,

$\mathbf{0}_\Sigma(x) = 0$ for all $x \in \Sigma$.

If A, B are sets of multisets, we define

$A \oplus B = \{\alpha \oplus \beta \,|\, \alpha \in A \text{ and } \beta \in B\}$.

Further, for any multisets $\alpha, \beta \in \Sigma^{\oplus}$, we define the *difference* $\alpha \ominus \beta$, the *union* $\alpha \sqcup \beta$, and the *inclusion* $\alpha \sqsubseteq \beta$ by

$(\alpha \ominus \beta)(x) = \max\{0, \alpha(x) - \beta(x)\}$ for all $x \in \Sigma$,

$(\alpha \sqcup \beta)(x) = \max\{\alpha(x), \beta(x)\}$ for all $x \in \Sigma$,

$\alpha \sqsubseteq \beta$ iff $\alpha(x) \leq \beta(x)$ for all $x \in \Sigma$.

We use the notation $\langle y \rangle$ for singleton multisets, i.e., $\langle y \rangle(x) = 0$ for $x \neq y$ and $\langle y \rangle(y) = 1$. Furthermore, we denote by $\Sigma^{\langle\rangle}$ the set of all singleton multisets over Σ. If $\alpha_i = \alpha \in \Sigma^{\oplus}$ for $i \in \{1, \dots, m\}$, we write α^m instead of $\alpha_1 \oplus \cdots \oplus \alpha_m$ and we mean $\alpha^0 = \mathbf{0}_\Sigma$. We use the notation $\bigoplus$ for repeated addition of multisets, i.e., $\bigoplus_{i=1}^{n} \alpha_i = \alpha_1 \oplus \cdots \oplus \alpha_n$. For a multiset α, we denote the number of occurrences of a symbol $a \in \Sigma$ in α by $|\alpha|_a$. By cardinality of a multiset α we understand card$(\alpha) = \sum_{a\in\Sigma} |\alpha|_a$. Since each multiset over an alphabet $\{a_1, \dots, a_k\}$ can be represented as a vector from $\mathbf{N}^k$, we can also write every multiset $\alpha = \langle a_1 \rangle^{i_1} \oplus \ldots \oplus \langle a_k \rangle^{i_k}$ with $i_1, \dots, i_k \in \mathbf{N}$ as the vector $\alpha = (i_1, \dots, i_k)$.

The interested reader can find more about multiset theory for example in [24] and [3].

The following definition of (regular) multiset grammar is based on [10] and [25]. There are few unimportant differences (like one vs more start symbols or slightly different demands on right hand sides of productions of a regular grammar) which do not influence its generative power.

Definition 1. A *multiset grammar* is an ordered quadruple $G = (N, \Sigma, P, S)$, where N and Σ are disjoint alphabets of *nonterminals* and *terminals*, respectively, $P \subseteq [N^{\langle\rangle} \oplus (N \cup \Sigma)^{\oplus}] \times (N \cup \Sigma)^{\oplus}$ is a finite set of *multiset rewriting productions*, and $S \in N$ is the *start symbol*. The productions (α, β) of P are usually denoted by $\alpha \to \beta$.

For $\mu_1, \mu_2 \in (N \cup \Sigma)^{\oplus}$, we write $\mu_1 \Rightarrow \mu_2$, if there exist $\alpha \to \beta \in P$ and $\gamma \in (N \cup \Sigma)^{\oplus}$ such that $\mu_1 = \gamma \oplus \alpha$, $\mu_2 = \gamma \oplus \beta$. We denote the reflexive and transitive closure of the relation $\Rightarrow$ by $\Rightarrow^*$. The *multiset language generated by* $G = (N, \Sigma, P, S)$ is defined by

$$M(G) = \{\omega \in \Sigma^{\oplus} \,|\, \langle S \rangle \Rightarrow^* \omega\}.$$

A multiset grammar $G = (N, \Sigma, P, S)$ is called *regular* if every production $\alpha \to \beta \in P$ has one of the following forms (with $A, B \in N$, $a, b \in \Sigma$):

- $\langle A \rangle \to \langle b \rangle \oplus \langle B \rangle$,
- $\langle A \rangle \to \langle a \rangle$,
- $\langle A \rangle \to \mathbf{0}_\Sigma$.

A multiset language generated by a regular multiset grammar is called *regular*.

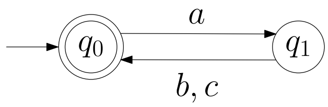

Fig. 1. Multiset finite automaton A.

Example 1. Consider the regular multiset grammar $G = (N, \Sigma, P, S)$ where

$$N = \{A, S\},$$
$$\Sigma = \{a, b, c\},$$
$$P = \big\{\langle S\rangle \to \langle a\rangle \oplus \langle A\rangle,\ \langle A\rangle \to \langle b\rangle \oplus \langle S\rangle,\ \langle A\rangle \to \langle c\rangle \oplus \langle S\rangle,\ \langle S\rangle \to \mathbf{0}_\Sigma\big\}.$$

Then we have, for example:

$$\langle S\rangle \Rightarrow \langle a\rangle \oplus \langle A\rangle \Rightarrow \langle a\rangle \oplus \langle b\rangle \oplus \langle S\rangle \Rightarrow \langle a\rangle^2 \oplus \langle b\rangle \oplus \langle A\rangle \Rightarrow \langle a\rangle^2 \oplus \langle b\rangle \oplus \langle c\rangle \oplus \langle S\rangle \Rightarrow$$
$$\Rightarrow \ldots \Rightarrow \langle a\rangle^5 \oplus \langle b\rangle^2 \oplus \langle c\rangle^3 \oplus \langle S\rangle \Rightarrow \langle a\rangle^5 \oplus \langle b\rangle^2 \oplus \langle c\rangle^3.$$

It is easy to see that

$$M(G) = \{\langle a\rangle^i \oplus \langle b\rangle^j \oplus \langle c\rangle^k \mid i = j + k\}.$$ □

Definition 2 ([11,16]). A *multiset finite automaton* is an ordered quintuple $A = (Q, \Sigma, \delta, q_0, F)$ where Q is a nonempty finite set of states, Σ is the input alphabet, q_0 is the initial state, $F \subseteq Q$ is the set of final states, and $\delta \subseteq Q \times \Sigma \times Q$ is the transition relation.

If the transition relation fulfils the following condition

- $(q, a, q') \in \delta$ and $(q, b, q'') \in \delta$ imply $a = b$ and $q' = q''$ for all $q, q', q'' \in Q,\ a, b \in \Sigma$,

then the automaton A is said to be a *deterministic multiset finite automaton*.

We extend the relation δ to relation $\delta^* \subseteq Q \times \Sigma^\oplus \times Q$ in the recursive way:

- $(q, \mathbf{0}_\Sigma, r) \in \delta^*$ iff $r = q$,
- $(q, \alpha, s) \in \delta^*$ if there are $r \in Q,\ a \in \Sigma$ such that $\langle a\rangle \sqsubseteq \alpha,\ (q, a, r) \in \delta$ and $(r, \alpha \ominus \langle a\rangle, s) \in \delta^*$.

The *multiset language* $M(A)$ accepted by the multiset finite automaton A is defined by

$$M(A) = \{\alpha \in \Sigma^\oplus \mid (q_0, \alpha, q) \in \delta^* \text{ for some } q \in F\}.$$

Otherwise stated, the multiset language $M(A)$ consists of all multisets α such that the automaton A starting its computation in q_0 with α on its 'input' finishes its work in a final state with $\mathbf{0}_\Sigma$ on its 'input'. Realize that computation of the automaton A does not depend on some strict order of symbols in the 'input multiset'.

Example 2. Consider the multiset finite automaton $A = (Q, \Sigma, \delta, q_0, F)$ where $Q = \{q_0, q_1\}$, $\Sigma = \{a, b, c\}$, $\delta = \{(q_0, a, q_1), (q_1, b, q_0), (q_1, c, q_0)\}$, and $F = \{q_0\}$. We can illustrate the automaton by its state diagram in Fig. 1.

It is easy to see that $M(A) = \{\langle a\rangle^i \oplus \langle b\rangle^j \oplus \langle c\rangle^k \mid i = j + k\}$ which is the same multiset language as in Example 1. □

Theorem 1 ([11]). *The family of multiset regular languages is equal to the family of multiset languages accepted by multiset finite automata.*

Contrary to the classical (i.e., string) Automata theory, deterministic multiset finite automata are less powerful than the nondeterministic ones as states the next theorem.

Theorem 2 ([11]). *The family of languages accepted by deterministic multiset finite automata is the proper subfamily of the family of languages accepted by multiset finite automata.*

The following example illustrates validity of Theorem 2.

Example 3 ([11]). Let $M = \big\{\big(\langle a\rangle \oplus \langle b\rangle^2\big)^m \oplus \big(\langle a\rangle^2 \oplus \langle b\rangle\big)^n \mid m, n > 1\big\}$. Obviously, the multiset language M is accepted by a multiset finite automaton (with two computational branches). Let $A = (Q, \{a, b\}, \delta, q_0, F)$ be a deterministic multiset finite automaton accepting M. Considering $m = 2, n = k + 1$ or $m = 2k, n = 2$, respectively, we obtain both $\langle a\rangle^{2k+4} \oplus \langle b\rangle^{k+5} \in M$ and $\langle a\rangle^{2k+4} \oplus \langle b\rangle^{4k+2} \in M$ for $k > 1$. Hence, there are final states $q_1, q_2 \in Q$ such that $(q_0, \langle a\rangle^{2k+4} \oplus \langle b\rangle^{k+5}, q_1) \in \delta^*$ and $(q_0, \langle a\rangle^{2k+4} \oplus \langle b\rangle^{4k+2}, q_2) \in \delta^*$. Realizing determinism of the computation, we have $(q_1, \langle b\rangle^{3(k-1)}, q_2) \in \delta^*$. Thus, computation starting in the state q_1, processing $\langle b\rangle^{3(k-1)}$ with sufficiently great k and finishing in the state q_2 means a cycling. If we denote positive length of one of the cycles by c, then $(q_1, \langle b\rangle^{3(k-1)+jc}, q_2) \in \delta^*$ and $\langle a\rangle^{2k+4} \oplus \langle b\rangle^{k+5+3(k-1)+jc} \in M(A)$ for $j, k > 1$. However, it can easily be checked that M does not contain multisets $\langle a\rangle^{2k+4} \oplus \langle b\rangle^{4k+2+jc}$ with $jc > 6$ which is a contradiction with the assumption $A(M) = M$. □

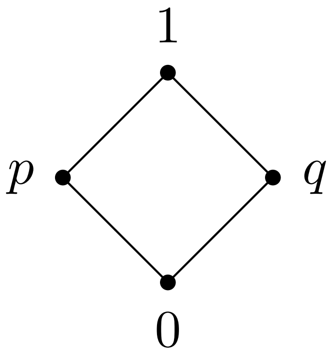

Fig. 2. Ordered set L.

3. Fuzzy multiset regular grammars and fuzzy multiset finite automata

As a set of truth values, we use a commutative integral quantale (see [26,27]), i.e., an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, 0, 1\rangle$ such that

- $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with the least element 0 and the greatest element 1,
- $\langle L, \otimes, 1\rangle$ is a commutative monoid with the neutral element 1,
- $0 \otimes a = 0$ for all $a \in L$,
- $a \otimes \left(\bigvee_{i\in I} b_i\right) = \bigvee_{i\in I}(a \otimes b_i)$ for any index set I and for all $a, b_i \in L$.

Recall that a *fuzzy set* A in a universe set X is any mapping $A : X \to L$, $A(x)$ being interpreted as the truth degree of the fact that 'x belongs to A' and being called *membership value*. A *fuzzy relation* R between sets X and Y is defined as a mapping $R : X \times Y \to L$. Analogously, a *fuzzy ternary relation* $\widetilde{R}$ is defined as a mapping $\widetilde{R} : X \times Y \times Z \to L$, etc. For any fuzzy set A in X, the set $\mathrm{supp}(A) = \{x \in X \mid A(x) > 0\}$ is called *support* of A. For $x \in X$ and $a \in L$, $\{x/a\}$ is the fuzzy set in X defined by $\{x/a\}(x) = a$ and $\{x/a\}(y) = 0$ for $y \neq x$. If $c_i = c \in L$ for $i \in \{1, \dots, m\}$, we write c^m instead of $c_1 \otimes \cdots \otimes c_m$ and we mean $c^0 = 1$.

The following definition of fuzzy multiset (regular) grammar is based on [15] which uses different notation and a simpler structure of truth values.

Definition 3. A *fuzzy multiset regular grammar* is an ordered quadruple $G = (N, \Sigma, P, S)$, where N and Σ are disjoint alphabets of *nonterminals* and *terminals*, respectively, $P : \left[N^{\langle\rangle} \times \left[\left(\Sigma^{\langle\rangle} \oplus N^{\langle\rangle}\right) \cup \Sigma^{\langle\rangle} \cup \{\mathbf{0}_\Sigma\}\right]\right] \to L$ is a fuzzy set of *productions*, and $S \in N$ is the *start symbol*. The productions $(\langle A\rangle, \alpha)$ of P are usually denoted by $\langle A\rangle \to \alpha$ and their membership value is denoted by $P(\langle A\rangle \to \alpha)$.

Further, we define the fuzzy relation $\Rightarrow$ between sets $N^{\langle\rangle} \oplus \Sigma^{\oplus}$ and $(N \cup \Sigma)^{\oplus}$ by

$$\Rightarrow(\alpha, \beta) = \bigvee_{\substack{A \in N \\ \beta' \in (N \cup \Sigma)^{\oplus}}} \left\{P(\langle A\rangle \to \beta') \mid \alpha = \gamma \oplus \langle A\rangle \text{ and } \beta = \gamma \oplus \beta' \text{ with } \gamma \in \Sigma^{\oplus}\right\}.$$

We proceed with iterations of the last fuzzy relation. We define for all $\alpha, \beta \in (N \cup \Sigma)^{\oplus}$:

$$\Rightarrow^0(\alpha, \beta) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{otherwise.} \end{cases}$$

$$\Rightarrow^n(\alpha, \beta) \overset{\text{def}}{=} \bigvee_{\gamma \in (N \cup \Sigma)^{\oplus}} \Rightarrow(\alpha, \gamma) \otimes \Rightarrow^{n-1}(\gamma, \beta) \quad \text{for all } n \geq 1.$$

$$\Rightarrow^*(\alpha, \beta) \overset{\text{def}}{=} \bigvee_{i \in \mathbf{N}} \Rightarrow^i(\alpha, \beta).$$

The *fuzzy multiset language generated by a fuzzy multiset regular grammar* $G = (N, \Sigma, P, S)$ is the fuzzy set $M(G)$ given by

$$M(G)(\gamma) = \begin{cases} \Rightarrow^*(\langle S\rangle, \gamma) & \text{for all } \gamma \in \Sigma^{\oplus}, \\ 0 & \text{for all } \gamma \in (N \cup \Sigma)^{\oplus} - \Sigma^{\oplus}. \end{cases}$$

and is called a *fuzzy multiset regular language*.

Example 4. Consider the structure of truth values $\mathbf{L} = \langle L, \wedge, \vee, \otimes, 0, 1\rangle$ where $L = \{0, p, q, 1\}$ is the ordered set with Hasse diagram depicted in Fig. 2. The operation $\otimes$ is given by the table in Fig. 3.

Let $G = (N, \Sigma, P, S)$ be a fuzzy multiset regular grammar where

$N = \{S, A\}$,

$\Sigma = \{a\}$,

$\otimes$	0	p	q	1
0	0	0	0	0
p	0	p	0	p
q	0	0	q	q
1	0	p	q	1

Fig. 3. Operation $\otimes$.

$$\mathrm{supp}(P) = \{\langle S\rangle \to \langle a\rangle,\ \langle S\rangle \to \langle a\rangle \oplus \langle A\rangle,\ \langle A\rangle \to \langle a\rangle \oplus \langle A\rangle,\ \langle A\rangle \to \mathbf{0}_\Sigma\}.$$

and

$$\begin{aligned} &P(\langle S\rangle \to \langle a\rangle) = p,\\ &P(\langle S\rangle \to \langle a\rangle \oplus \langle A\rangle) = P(\langle A\rangle \to \mathbf{0}_\Sigma) = q,\\ &P(\langle A\rangle \to \langle a\rangle \oplus \langle A\rangle) = 1. \end{aligned}$$

We have

$$\begin{aligned} \Rightarrow^0 (\langle S\rangle, \mathbf{0}_\Sigma) &= 0,\\ \Rightarrow^1 (\langle S\rangle, \langle a\rangle) &= \left[\Rightarrow (\langle S\rangle, \langle a\rangle)\right] \otimes \left[\Rightarrow^0 (\langle a\rangle, \langle a\rangle)\right] = P(\langle S\rangle \to \langle a\rangle) \otimes 1 = p \otimes 1 = p,\\ \Rightarrow^2 (\langle S\rangle, \langle a\rangle) &= \left[\Rightarrow (\langle S\rangle, \langle a\rangle \oplus \langle A\rangle)\right] \otimes \left[\Rightarrow (\langle a\rangle \oplus \langle A\rangle, \langle a\rangle \oplus \mathbf{0}_\Sigma)\right] = P(\langle S\rangle \to \langle a\rangle \oplus \langle A\rangle) \otimes P(\langle A\rangle \to \mathbf{0}_\Sigma) = q \otimes q = q,\\ \Rightarrow^* (\langle S\rangle, \langle a\rangle) &= \left[\Rightarrow^1 (\langle S\rangle, \langle a\rangle)\right] \vee \left[\Rightarrow^2 (\langle S\rangle, \langle a\rangle)\right] = p \vee q = 1,\\ \Rightarrow^3 \left(\langle S\rangle, \langle a\rangle^2\right) &= \left[\Rightarrow (\langle S\rangle, \langle a\rangle \oplus \langle A\rangle)\right] \otimes \left[\Rightarrow (\langle a\rangle \oplus \langle A\rangle, \langle a\rangle^2 \oplus \langle A\rangle)\right] \otimes \left[\Rightarrow (\langle a\rangle^2 \oplus \langle A\rangle, \langle a\rangle^2)\right] =\\ &= P(\langle S\rangle \to \langle a\rangle \oplus \langle A\rangle) \otimes P(\langle A\rangle \to \langle a\rangle \oplus \langle A\rangle) \otimes P(\langle A\rangle \to \mathbf{0}_\Sigma) = q \otimes 1 \otimes q = q,\\ \Rightarrow^4 \left(\langle S\rangle, \langle a\rangle^3\right) &= \ldots = q \otimes 1 \otimes 1 \otimes q = q,\\ &\text{etc.} \end{aligned}$$

It is easy to see that

$$M(G)(\langle a\rangle^m) = \begin{cases} 0 & \text{if } m = 0,\\ 1 & \text{if } m = 1,\\ q & \text{if } m > 1. \end{cases}$$ □

Definition 4 ([16,28])**.** A *fuzzy multiset finite automaton* (FMFA) is an ordered quintuple $A = (Q, \Sigma, \delta, q_0, F)$ where Q is a nonempty finite set of states, Σ is the input alphabet, q_0 is the initial state, $F : Q \to L$ is a fuzzy set in Q, and $\delta : Q \times \Sigma \times Q \to L$ is the fuzzy transition relation. A state $q \in Q$ is said to be a *final state* of A if $F(q) > 0$.

If the fuzzy transition relation fulfils the following condition

- $\delta(q, a, q') > 0$ and $\delta(q, b, q'') > 0$ imply $a = b$ and $q' = q''$ for all $q, q', q'' \in Q,\ a, b \in \Sigma$,

then the automaton A is said to be a *deterministic fuzzy multiset finite automaton*.

We extend the fuzzy relation δ to fuzzy relation $\delta^* : Q \times \Sigma^\oplus \times Q \to L$ in the following way.

$$\delta^*(q, \mathbf{0}_\Sigma, r) = \begin{cases} 1 & \text{if } r = q,\\ 0 & \text{otherwise,} \end{cases}$$

$$\delta^*(q, \alpha, s) = \bigvee_{\substack{r_0, \ldots, r_k \in Q\\ r_0 = q,\ r_k = s\\ a_1, \ldots, a_k \in \Sigma\\ \langle a_1\rangle \oplus \cdots \oplus \langle a_k\rangle = \alpha}} \{\delta(r_0, a_1, r_1) \otimes \cdots \otimes \delta(r_{k-1}, a_k, r_k)\} \text{ for all } \alpha \in \Sigma^\oplus, \mathrm{card}(\alpha) = k > 0.$$

The *fuzzy multiset language* $M(A)$ accepted by the FMFA A is defined by

$$M(A)(\alpha) = \bigvee_{q \in Q} \left\{\delta^*(q_0, \alpha, q) \otimes F(q)\right\} \text{ for all } \alpha \in \Sigma^\oplus \tag{1}$$

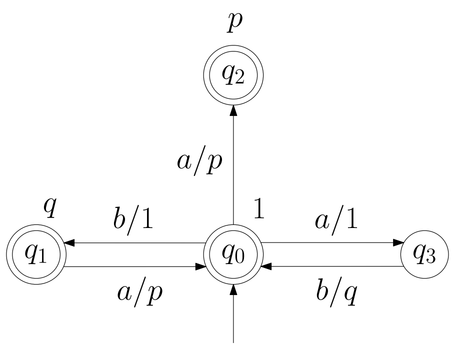

Fig. 4. FMFA B.

and is called an FMFA–*language*.[1,2]

A modification of FMFA consisting in fuzzy transition relation $\delta : Q \times \Sigma^{\oplus} \times Q \to L$ with finite support is called *FMFA with extended transition relation*.

We extend the fuzzy relation δ of FMFA with extended transition relation to fuzzy relation $\delta^* : Q \times \Sigma^{\oplus} \times Q \to L$ in the following way.

$$\delta^*(q, \mathbf{0}_\Sigma, r) = \delta(q, \mathbf{0}_\Sigma, r) \text{ for all } r, q \in Q,$$
$$\delta^*(q, \alpha, s) = \bigvee_{\substack{\beta \in \Sigma^{\oplus} \\ \beta \sqsubseteq \alpha \\ r \in Q}} \{\delta(q, \beta, r) \otimes \delta^*(r, \alpha \ominus \beta, s)\} \text{ for all } \alpha \in \Sigma^{\oplus}, r, q \in Q.$$

The fuzzy multiset language $M(A)$ accepted by the FMFA A with extended transition relation is defined by (1) as in the previous case. FMFAs A and B (of any kind) are called *equivalent* iff $M(A) = M(B)$.

Theorem 3. *Any fuzzy multiset language is accepted by an FMFA if and only if it is accepted by an FMFA with extended transition relation.*

Proof. The direction from FMFA to FMFA with extended transition relation is straightforward, the reverse direction is a slight generalization of Theorem 4.2 from [15]. □

Example 5. Consider the structure of truth values $\mathbf{L} = \langle L, \wedge, \vee, \otimes, 0, 1\rangle$ from Example 4, i.e., $L = \{0, p, q, 1\}$ is the ordered set with Hasse diagram depicted in Fig. 2. The operation $\otimes$ is given by the table in Fig. 3. Let $B = (Q, \Sigma, \delta, q_0, F)$ be an FMFA with

$$Q = \{q_0, q_1, q_2, q_3\},$$
$$\Sigma = \{a, b\},$$
$$\delta(q_0, a, q_2) = p,$$
$$\delta(q_0, a, q_3) = 1,$$
$$\delta(q_0, b, q_1) = 1,$$
$$\delta(q_1, a, q_0) = p,$$
$$\delta(q_3, b, q_0) = q,$$
$$\delta(q_i, x, q_j) = 0 \text{ otherwise},$$
$$F(q_0) = 1,\ F(q_1) = q,\ F(q_2) = p,\ F(q_3) = 0.$$

We can illustrate the fuzzy automaton more lucidly in Fig. 4 with the following adaptation of classical state diagram (see, e.g., [29] or [16]): nodes of the labelled directed graph represent states of the automaton, the initial state is indicated by the arrow pointing at it from nowhere, each final state q is depicted by double circle with the outer label $F(q)$, and each arc in the graph coincides with a non-null transition (if the arc goes from state q to state r and $\delta(q, a, r) = \mu$ then the arc is labelled by a/μ).

We have, for example,

$$\delta^*(q_0, \langle a\rangle \oplus \langle b\rangle, q_0) = [\delta(q_0, a, q_3) \otimes \delta(q_3, b, q_0)] \vee [\delta(q_0, b, q_1) \otimes \delta(q_1, a, q_0)] = [1 \otimes q] \vee [1 \otimes p] = q \vee p = 1,$$
$$\delta^*(q_0, \langle a\rangle^2 \oplus \langle b\rangle, q_2) = [\delta(q_0, a, q_3) \otimes \delta(q_3, b, q_0) \otimes \delta(q_0, a, q_2)] \vee [\delta(q_0, b, q_1) \otimes \delta(q_1, a, q_0) \otimes \delta(q_0, a, q_2)]$$

[1] Note that time complexity of evaluating $M(A)(\alpha)$ depends not only on searching nondeterministic computation tree of A (which requires an exponential time) but also on complexity of determining values of the operation $\otimes$ and the supremum which can vary depending on the considered structure of truth values.

[2] If A is deterministic and $M(A)(\alpha) > 0$ then there is a unique sequence $q_0, \dots, q_k$ such that $\delta^*(q_0, \alpha, q) = \delta(q_0, a_1, q_1) \otimes \cdots \otimes \delta(q_{k-1}, a_k, q_k)$ with $\langle a_1\rangle \oplus \cdots \oplus \langle a_k\rangle = \alpha$. Here, time complexity of evaluating $M(A)(\alpha)$ depends again on complexity of determining values of the operation $\otimes$.

$$\begin{aligned}
&=[1 \otimes q \otimes p] \vee [1 \otimes p \otimes p] = 0 \vee p = p,\\
\delta^*(q_0, \langle a\rangle^2 \oplus \langle b\rangle, q_3) &=[\delta(q_0, a, q_3) \otimes \delta(q_3, b, q_0) \otimes \delta(q_0, a, q_3)] \vee [\delta(q_0, b, q_1) \otimes \delta(q_1, a, q_0) \otimes \delta(q_0, a, q_3)]\\
&=[1 \otimes q \otimes 1] \vee [1 \otimes p \otimes 1] = q \vee p = 1,\\
\delta^*(q_0, \langle a\rangle \oplus \langle b\rangle^2, q_1) &=[\delta(q_0, a, q_3) \otimes \delta(q_3, b, q_0) \otimes \delta(q_0, b, q_1)] \vee [\delta(q_0, b, q_1) \otimes \delta(q_1, a, q_0) \otimes \delta(q_0, b, q_1)]\\
&=[1 \otimes q \otimes 1] \vee [1 \otimes p \otimes 1] = q \vee p = 1.
\end{aligned}$$

Clearly,

$$\delta^*(q_0, \langle a\rangle \oplus \langle b\rangle, q_i) = \begin{cases} 1 & \text{if } i = 0,\\ 0 & \text{otherwise.}\end{cases}$$

$$\delta^*(q_0, \langle a\rangle^2 \oplus \langle b\rangle, q_i) = \begin{cases} p & \text{if } i = 2,\\ 1 & \text{if } i = 3,\\ 0 & \text{otherwise.}\end{cases}$$

$$\delta^*(q_0, \langle a\rangle \oplus \langle b\rangle^2, q_i) = \begin{cases} 1 & \text{if } i = 1,\\ 0 & \text{otherwise.}\end{cases}$$

Hence,

$$\begin{aligned}
M(B)(\langle a\rangle \oplus \langle b\rangle) &=[\delta^*(q_0, \langle a\rangle \oplus \langle b\rangle, q_0) \otimes F(q_0)] = 1 \otimes 1 = 1,\\
M(B)(\langle a\rangle^2 \oplus \langle b\rangle) &=[\delta^*(q_0, \langle a\rangle^2 \oplus \langle b\rangle, q_2) \otimes F(q_2)] \vee [\delta^*(q_0, \langle a\rangle^2 \oplus \langle b\rangle, q_3) \otimes F(q_3)] = [p \otimes p] \vee [1 \otimes 0] = p \vee 0 = p,\\
M(B)(\langle a\rangle \oplus \langle b\rangle^2) &=[\delta^*(q_0, \langle a\rangle \oplus \langle b\rangle^2, q_1) \otimes F(q_1)] = 1 \otimes q = q,
\end{aligned}$$

It is easy to see that

$$M(B)(\alpha) = \begin{cases} 1 & \text{if } |\alpha|_a = |\alpha|_b,\\ p & \text{if } |\alpha|_a = |\alpha|_b + 1,\\ q & \text{if } |\alpha|_b = |\alpha|_a + 1,\\ 0 & \text{otherwise.}\end{cases}$$

□

Some notes to the definition of FMFA

Our definition of FMFA differs from the one of [15] in several points: we use a single initial state (instead of a fuzzy set of initial states) and a fuzzy transition relation related to a symbol from Σ instead of multisets over Σ. Despite of these differences, none of them influences computational power of the automata as follows from Theorems 4.1 and 4.2 of [15].

What is more crucial is that due to an oversight, the fuzzy transition relation does not have finite support in [15] because it uses the infinite set $\Sigma^{\oplus}$ of all multiset over Σ.

The necessity for the fuzzy transition relation to have finite support (like in our definition of FMFA with extended transition relation) is well-known from (classical) Formal languages theory: An automaton with infinite transition relation δ can accept any language L_A as confirms the setting $\delta = \{(q_0, w, g_f) \mid w \in L_A, q_f \text{ is a final state}\}$.

Unfortunately, the wrong definition (with missing demand of finite support of the fuzzy transition relation) appears not only in [15] but also in many papers of its followers which can lead to some incorrect results in their papers. (Especially, when their proofs use infinite sets where finite ones are expected.)

A note to deterministic FMFA

It should be also mentioned that many papers (see, e.g., [17,18]) use a notion of 'deterministic FMFA' which does not correspond to the general perception of deterministic computation as computation without any branching. Namely, transition relation of their 'deterministic FMFA' must satisfy Condition C1: For every pair $(q, \alpha) \in Q \times \Sigma^{\oplus}$, there is at most one q' such that $\delta(q, \alpha, q') > 0$. Insufficiency of the condition for ensuring determinism is illustrated in the following example.

Example 6. Let $C = (Q, \Sigma, \delta, q_0, F)$ be an FMFA with extended transition relation and with state diagram from Fig. 5.

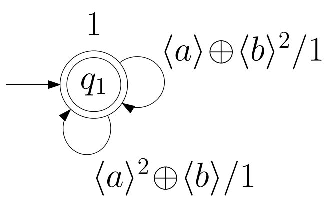

Fig. 5. FMFA C with extended transition relation.

Obviously, $M(A)(\alpha) = 1$ iff $\alpha \in \left\{ \left(\langle a \rangle \oplus \langle b \rangle^2 \right)^m \oplus \left(\langle a \rangle^2 \oplus \langle b \rangle \right)^n \mid m, n \geq 0 \right\}$ (see also Example 3). Clearly, processing $\alpha = \langle a \rangle^3 \oplus \langle b \rangle^3$ can start either with use of transition $(q_0, \langle a \rangle \oplus \langle b \rangle^2, q_0)$ or $(q_0, \langle a \rangle^2 \oplus \langle b \rangle, q_0)$. So, the computation is nondeterministic despite the fact that the transition relation satisfies Condition C1. □

Example 6 justifies the opinion that more appropriate name for an FMFA with extended transition relation satisfying Condition C1 is 'pseudodeterministic FMFA'. Similarly, Example 6 points also at incorrect use of the adjective deterministic in [19] with differently formulated 'deterministic FMFA'.[3] In any case, these pseudodeterministic automata must not be confused with deterministic FMFAs of this paper (whose computations never branch) and which have smaller computational power than FMFAs as is described in Theorem 5.

Theorem 4. *The family of fuzzy multiset languages accepted by FMFAs is equal to the family of all fuzzy multiset regular languages.*

Proof. Standard techniques known from (crisp) Formal languages theory can be used to prove the equality (see also [15] with more specific structure of truth values). □

As a direct consequence of Theorem 2, we obtain the next theorem.

Theorem 5 ([16]). *The family of languages accepted by deterministic fuzzy multiset finite automata is the proper subfamily of the family of languages accepted by fuzzy multiset finite automata.*

4. Regular operations on fuzzy multiset languages and fuzzy multiset regular expressions

Definition 5 ([15]). Let Σ be an alphabet and $M, M_1, M_2 : \Sigma^{\oplus} \to L$. We define the *addition* $M_1 \oplus M_2$, the *union* $M_1 \cup M_2$, and the *iteration*[4] $M^{\oplus}$ by

$$(M_1 \oplus M_2)(\gamma) = \bigvee_{\substack{\alpha, \beta \in \Sigma^{\oplus} \\ \alpha \oplus \beta = \gamma}} \left\{ M_1(\alpha) \otimes M_2(\beta) \right\} \text{ for all } \gamma \in \Sigma^{\oplus},$$

$$(M_1 \cup M_2)(\gamma) = M_1(\gamma) \vee M_2(\gamma) \text{ for all } \gamma \in \Sigma^{\oplus},$$

$$M^{\oplus} = M^0 \cup M \cup M^2 \cup \dots$$

where

$$M^0(\alpha) = \begin{cases} 1 & \text{if } \alpha = \mathbf{0}_\Sigma, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$M^i = M \oplus M^{i-1} \text{ for all } i \geq 2.$$

The next theorem was formulated and proved in [15] for more specific structure of truth values ($[0,1]$ with operation $\otimes = \min$) and on the basis of a little bit shifted definition of FMFA. Adapting the proof to our more general structure of truth values is straightforward.

Theorem 6. *The family of fuzzy multiset regular languages is closed under the operations of union, addition, and iteration.*

The following definition generalizes the classical notion of a regular expression (see, e.g., [21,30]) in a straightforward way (see also [31] or [32]).

Definition 6. Let Σ, $Z = \{\emptyset, \mathbf{0}_\Sigma, \cup, ^{\oplus}, (,)\}$, and L be disjoint alphabets. A sequence ρ of symbols over $\Sigma \cup Z \cup L$ is called a fuzzy multiset regular expression if ρ is one of the following forms:

1. $\emptyset$,
2. $\mathbf{0}_\Sigma a$ where $a \in L - \{0\}$,
3. xa where $x \in \Sigma$, $a \in L - \{0\}$,
4. $(\rho_1 \cup \rho_2)$ where ρ_1, ρ_2 are fuzzy multiset regular expressions,
5. $(\rho_1 \rho_2)$ where ρ_1, ρ_2 are fuzzy multiset regular expressions,
6. $\left(\rho_1^{\oplus}\right)$ where ρ_1 is a fuzzy multiset regular expression.

Every fuzzy multiset regular expression ρ describes the fuzzy multiset language $M(\rho)$ according to the following conventions:

1. $M(\emptyset) = \emptyset$,
2. $M(\mathbf{0}_\Sigma a) = \left\{ \mathbf{0}_\Sigma / a \right\}$ where $a \in L - \{0\}$,
3. $M(xa) = \{\langle x \rangle / a\}$ where $x \in \Sigma$, $a \in L - \{0\}$,
4. $M(\rho_1 \cup \rho_2) = M(\rho_1) \cup M(\rho_2)$ where ρ_1, ρ_2 are fuzzy multiset regular expressions,

[3] Note that proofs of equality between the family of fuzzy multiset languages accepted by FMFAs and the family of fuzzy multiset languages accepted by pseudodeterministic FMFAs use incorrectly automata with infinitely many states in [19] and [17]. A proof of an analogous statement in [18] depends on infinitely many multisets of $\Sigma^{\oplus}$ in transition relation.

[4] In [15], the iteration operation is called closure operation.

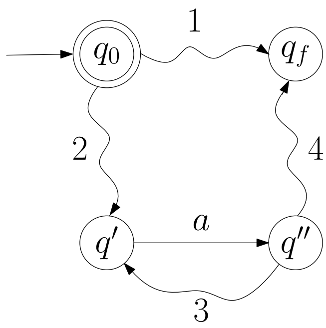

Fig. 6. FMFA A.

5. $M(\rho_1\rho_2) = M(\rho_1) \oplus M(\rho_2)$ where ρ_1, ρ_2 are fuzzy multiset regular expressions,
6. $M\left(\rho_1^{\oplus}\right) = \left(M(\rho_1)\right)^{\oplus}$ where ρ_1 is a fuzzy multiset regular expression.

Example 7. Let [0; 1] be the closed interval of real numbers between 0 and 1 and let $\otimes$ denote minimum (i.e., $a \otimes b = \min\{a, b\}$). Consider $\Sigma = \{c, d\}$ and fuzzy multiset regular expression $\rho = c\,1(d\,1\,c\,1)^{\oplus} \cup d\,0.3(c\,0.4\,d\,0.5)^{\oplus}$ where unnecessary parentheses are omitted. It is easy to see that

$$M(\rho)(\alpha) = \begin{cases} 1 & \text{if } \alpha = \langle c\rangle^{n+1} \oplus \langle d\rangle^{n} \text{ with } n \geq 0, \\ 0.3 & \text{if } \alpha = \langle c\rangle^{n} \oplus \langle d\rangle^{n+1} \text{ with } n \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 7. *A fuzzy multiset language is accepted by an FMFA if and only if it is described by a fuzzy multiset regular expression.*

Proof. Let M be a fuzzy multiset language over an alphabet Σ.

A) Assume that there is a fuzzy multiset regular expression ρ such that $M = M(\rho)$.

1. If $\rho = \emptyset$ then $M = M(A)$ where $A = (Q, \Sigma, \delta, q_0, F)$ is an FMFA with $F : Q \to \{0\}$.
2. If $\rho = \mathbf{0}_\Sigma/a$ with $a \in L - \{0\}$ then $M(\rho) = \left\{\mathbf{0}_\Sigma/a\right\} = M(A)$ where $A = (\{q_0\}, \Sigma, \delta, q_0, \{q_0/a\})$ with δ defined by $\delta(q_0, x, q_0) = 0$ for all $x \in \Sigma$.
3. If $\rho = xa$ with $x \in \Sigma$ and $a \in L - \{0\}$ then $M(\rho) = \{\langle x\rangle/a\} = M(A)$ where $A = (\{q_0, q_1\}, \Sigma, \delta, q_0, \{q_1/a\})$ with δ defined for all $y \in \Sigma$ by

$$\delta(q_i, y, q_j) = \begin{cases} 1 & \text{if } i = 0, j = 1, \text{ and } y = x, \\ 0 & \text{otherwise.} \end{cases}$$

4. If ρ is equal to $(\rho_1 \cup \rho_2)$, $(\rho_1\rho_2)$ or $\left(\rho_1^{\oplus}\right)$ where ρ_1, ρ_2 are fuzzy multiset regular expressions, then $M(\rho)$ is accepted by a fuzzy multiset finite automaton by Theorem 6.

B) Let M be accepted by an FMFA $A = (Q, \Sigma, \delta, q_0, F)$ with n non-null transitions (where $n \in \mathbf{N}$), i.e., $\text{supp}(\delta) = n$.

We will use an approach from [33] (used for an alternative proof of Kleene's theorem) and prove existence of a fuzzy multiset regular expression ρ such that $M(\rho) = M$ by induction on n.

1) Let $n = 0$. Then $M = \emptyset$ or $M = \left\{\mathbf{0}_\Sigma/F(q_0)\right\}$ where these fuzzy multiset languages are described by fuzzy multiset regular expressions $\emptyset$ or $\mathbf{0}_\Sigma/F(q_0)$, respectively.

2) Let the fuzzy multiset regular expression ρ exist for any index from the set $\{0, \dots, n\}$ where $n \geq 0$. We will verify its existence for $n + 1$.

Assume that $A = (Q, \Sigma, \delta, q_0, F)$ contains $n + 1$ non-null transitions. Choose one of them, e.g. $\delta(q', a, q'')$ and transform δ to $\delta' : Q \times \Sigma \times Q \to L$ by setting $\delta'(q', a, q'') = 0$. Formally, δ' is defined for all $q, r \in Q$, $x \in \Sigma$ by

$$\delta'(q, x, r) = \begin{cases} 0 & \text{if } q = q', r = q'', \text{ and } x = a, \\ \delta(q, x, r) & \text{otherwise.} \end{cases}$$

Further, define 'smaller' fuzzy multiset finite automata (a rough illustration is in Fig. 6)

- $A_1 = (Q, \Sigma, \delta', q_0, F)$,
- $A_2 = (Q, \Sigma, \delta', q_0, \left\{q'/1\right\})$,
- $A_3 = (Q, \Sigma, \delta', q'', \left\{q'/1\right\})$,
- $A_4 = (Q, \Sigma, \delta', q'', F)$.

With regard to Definition 5, it is easy to see that

$$M(A) = M(A_1) \cup \Big[M(A_2) \oplus \{a/\delta(q', a, q'')\} \oplus \big(M(A_3) \oplus \{a/\delta(q', a, q'')\}\big)^{\oplus} \oplus M(A_4) \Big].$$

Since $A_1, \dots, A_4$ contain at most n non-null transitions, $M(A_1), \dots, M(A_4)$ are described by fuzzy multiset regular expressions according to induction hypothesis. Taking Definition 6 into account and realizing that $\{a/\delta(q', a, q'')\}$ is described by the fuzzy multiset regular expression $a\delta(q', a, q'')$, we obtain that $M(A)$ is described by a fuzzy multiset regular expression. So, the theorem holds true. □

5. Fuzzy semilinear sets

We begin with crisp linear and semilinear sets. The next definition follows from [34] and [35].

Definition 7. A multiset language M over an alphabet Σ is said to be *linear* if there are a multiset $\gamma_0 \in \Sigma^{\oplus}$ (called *constant*) and a finite set $\Gamma = \{\gamma_1, \dots, \gamma_m\} \subseteq \Sigma^{\oplus}$ of multisets (called *periods*) with $m \in \mathbf{N}$ such that

$$M = \left\{ \gamma_0 \oplus \bigoplus_{j=1}^{m} \gamma_j^{l_j} \mid l_1, \dots, l_m \in \mathbf{N} \right\}.$$

Note that $M = \{\gamma_0\} \oplus \bigoplus_{j=1}^{m} \{\gamma_j\}^{\oplus} = \{\gamma_0\} \oplus \Gamma^{\oplus}$.

A multiset language M over an alphabet Σ is said to be *semilinear* if it is a union of linear multiset languages. I.e., there are multisets $\gamma_{i,0}, \dots, \gamma_{i,m_i} \in \Sigma^{\oplus}$ and sets of periods $\Gamma_i = \{\gamma_{i,0}, \dots, \gamma_{i,m_i}\} \subseteq \Sigma^{\oplus}$ with $i \in \{1, \dots, n\}$, $n, m_i \in \mathbf{N}$ such that

$$M = \bigcup_{i=1}^{n} \left\{ \gamma_{i,0} \oplus \bigoplus_{j=1}^{m_i} \gamma_{i,j}^{l_{i,j}} \mid l_{i,1}, \dots, l_{i,m_i} \in \mathbf{N} \right\}.$$

Thus $M = \bigcup_{i=1}^{n} \big(\{\gamma_{i,0}\} \oplus (\Gamma_i)^{\oplus}\big)$.

As a special case of Proposition 1 from [11], we obtain the following theorem.

Theorem 8. *A multiset language M is accepted by an MFA if and only if it is semilinear.*

Fuzzification of the notions of linear and semilinear sets is straightforward.

Definition 8. A fuzzy multiset language M over an alphabet Σ is said to be expressed in *linear form* if

$$M = \left\{ \left\{ \frac{\gamma_0}{c_0} \right\} \oplus \bigoplus_{j=1}^{m} \left\{ \frac{\gamma_j}{c_j} \right\}^{\oplus} \right\}$$

for some $m \in \mathbf{N}, \gamma_0, \dots, \gamma_m \in \Sigma^{\oplus}, c_0, \dots, c_m \in L$.

A fuzzy multiset language M over an alphabet Σ is said to be expressed in *semilinear form* if it is written as a union of fuzzy multiset languages in linear form. I.e.,

$$M = \bigcup_{i=1}^{n} \left\{ \left\{ \frac{\gamma_{i,0}}{c_{i,0}} \right\} \oplus \bigoplus_{j=1}^{m_i} \left\{ \frac{\gamma_{i,j}}{c_{i,j}} \right\}^{\oplus} \right\}$$

for some $n, m_i \in \mathbf{N}, c_{i,j} \in L, \gamma_{i,j} \in \Sigma^{\oplus}$ with $j \in \{0, \dots, m_i\}, i \in \{1, \dots, n\}$.

Note that $\gamma \in \mathrm{supp}(M)$ can be expressed with help of different $\gamma_{i,0}, \gamma_{i,j}$ (for example, $\gamma = \langle a\rangle \oplus \langle b\rangle^3 \oplus \langle c\rangle^2 \in \{\langle a\rangle\} \oplus \{\langle b\rangle\}^{\oplus} \oplus \{\langle c\rangle\}^{\oplus}$ or $\gamma \in \{\langle a\rangle \oplus \langle b\rangle\} \oplus \{\langle b\rangle \oplus \langle c\rangle\}^{\oplus}$), but the value of $M(\gamma)$ is uniquely determined by Definition 5.

The next theorem states that a fuzzy multiset language M is regular if and only if it can be expressed in semilinear form. Note that in (non-multiset) regular languages, such an equality is being proved with help of deterministic finite automata because non-deterministic and deterministic finite automata have the same computational power. However, at multiset regular languages, the situation is different because nondeterministic multiset finite automata have bigger computational power than deterministic multiset finite automata (see Theorems 2 and 5).

Theorem 9. *A fuzzy multiset language M is accepted by an FMFA if and only if it can be expressed in semilinear form.*

Proof. 1) Since a fuzzy multiset language in semilinear form is defined as a union of languages in linear form and the family of fuzzy multiset regular languages is closed with respect to the operation of union by Theorem 6, it suffices to find an FMFA accepting a fuzzy multiset language M in linear form. So, assume $M = \left\{ \left\{ \frac{\gamma_0}{c_0} \right\} \oplus \bigoplus_{j=1}^{m} \left\{ \frac{\gamma_j}{c_j} \right\}^{\oplus} \right\}$ for some $m \in \mathbf{N}, \gamma_0, \dots, \gamma_m \in \Sigma^{\oplus}, c_0, \dots, c_m \in L$. Consider an FMFA $A = (\{q_0, q_1\}, \Sigma, \delta, q_0, \{q_1/1\})$ with extended transition relation where δ is defined by

$$\delta(q_i, \alpha, q_k) = \begin{cases} c_0 & \text{if } i = 0,\ k = 1 \text{ and } \alpha = \gamma_0, \\ c_j & \text{if } i = 1,\ k = 1 \text{ and } \alpha = \gamma_j \text{ with } 1 \le j \le m, \\ 0 & \text{otherwise.} \end{cases}$$

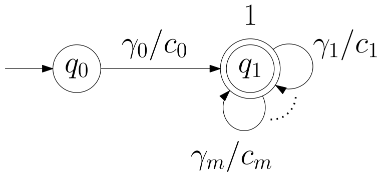

Fig. 7. FMFA A with extended transition relation.

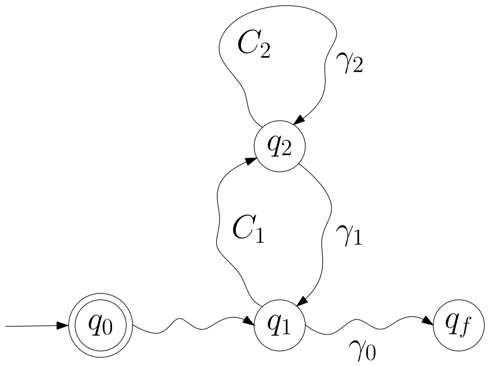

Fig. 8. MFA B.

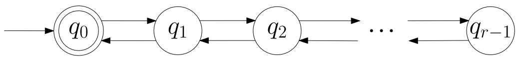

Fig. 9. State diagram of MFA C.

(Its state diagram is shown in Fig. 7.) It is easy to see that A accepts M.

Clearly, $M(A) = M$ which ensures together with Theorem 3 validity of one of the implications.

2) For easier insight into the reverse implication, we describe the basic idea on (crisp) MFA and its state diagram, i.e., a labelled directed graph. To relate the graph to the corresponding multiset language in semilinear form, it seems that connecting 'accepting directed paths' (i.e., directed paths from the initial state to a final state) to constants and all directed cycles to periods can work. However, consider automaton B from Fig. 8 where q_0 is the initial state, q_f is a final state, use of the directed path leading from q_0 to q_f via q_1 means processing γ_0, use of directed cycles C_1, C_2, and $C_1 + C_2$ means processing γ_1, γ_2, and $\gamma_1 \oplus \gamma_2$, respectively. Clearly, the multiset language $M' = \left\{\gamma_0 \oplus \gamma_1^i \oplus \gamma_2^j \oplus (\gamma_1 \oplus \gamma_2)^k \mid i, j, k \in \mathbf{N}\right\}$ is not accepted by MFA B because $\gamma_0 \oplus \gamma_2 \notin M(B)$. Therefore, directed walks (instead of directed paths[5]) of length up to $(r-1)^2$ (where the automaton has r states) will be used for constants in the semilinear form. The length $(r-1)^2$ is big enough (see, for example, automaton C with state diagram of the form from Fig. 9 where $r \geq 1$).

Now, let M be a fuzzy multiset language which is accepted by an FMFA $A = (Q, \Sigma, \delta, q_0, F)$ with r states. Consider its state diagram (i.e., a labelled directed graph). The graph contains a finite number of directed cycles $C_1, \ldots, C_s$. Let $W_1(q_{f_1}), \ldots, W_n(q_{f_n})$ be all directed walks in the graph such that they start in the node labelled q_0, end in a node labelled by a final state q_{f_i} (i.e., $F(q_{f_i}) > 0$), and have length at most $(r-1)^2$.

For every $W_i(q_{f_i})$ in the graph with edges labelled by $a_1/c_1, \ldots, a_p/c_p$ where p is length of the walk $W_i(q_{f_i})$, denote $\gamma(W_i(q_{f_i})) = \langle a_1 \rangle \oplus \cdots \oplus \langle a_p \rangle$ and $c(W_i(q_{f_i})) = c_1 \otimes \cdots \otimes c_p \otimes F(q_{f_i})$. For every cycle C_j in the graph with edges labelled by $b_1/d_1, \ldots, b_s/d_s$ where s is length of the cycle, denote $\gamma(C_j) = \langle b_1 \rangle \oplus \cdots \oplus \langle b_s \rangle$ and $c(C_j) = c_1 \otimes \cdots \otimes c_s$.

For each of the walks $W_i(q_{f_i})$, let $C_{i_1}, \ldots, C_{i_k}$ be the cycles which have a common vertex with $W_i(q_{f_i})$. Then every computation finishing in a final state q_f, which uses $W_i(q_f)$ and its related cycles, processes every multiset $\gamma = \gamma(W_i(q_f)) \oplus \bigoplus_{j=i_1}^{i_k} \gamma(C_j^{l_j})$ with $l_j \in \mathbf{N}$ where membership value of the computation is $c(W_i(q_f)) \otimes F(q_f) \otimes \bigotimes_{j=i_1}^{i_k} c(C_j^{l_j})$.

If $\gamma \in \Sigma^{\oplus}$ is arbitrary then

$$M(A)(\gamma) = \bigvee_{q \in Q} \left\{\delta^*(q_0, \gamma, q) \otimes F(q)\right\} = \bigvee_{\substack{q_f \in Q \\ F(q_f) > 0}} \left\{\delta^*(q_0, \gamma, q_f) \otimes F(q_f)\right\}. \tag{2}$$

[5] Recall that walk in a graph can use a node more than once while path cannot.

Since $\delta^*(q_0,\gamma,q_f)$ in Equality 2 correspond to 'accepting computations', there are $W_i(q_f)$ and $l_j \in \mathbf{N}$ such that $\gamma = \gamma(W_i(q_f)) \oplus \bigoplus_{j=i_1}^{i_k} \gamma(C_j^{l_j})$ (denote the set of all such i by I_γ) and

$$\bigvee_{\substack{q_f \in Q \\ F(q_f)>0}} \{\delta^*(q_0,\gamma,q_f) \otimes F(q_f)\} = \bigvee_{\substack{i \in I_\gamma \\ q_f \in Q \\ F(q_f)>0}} \left\{ c(W_i(q_f)) \otimes F(q_f) \otimes \bigotimes_{j=i_1}^{i_k} c(C_j^{l_j}) \right\}. \tag{3}$$

Note that $I = \bigcup\{I_\gamma \mid \gamma \in \Sigma^\oplus\}$ is finite because there is only finite number of $W_i(q_{f_i})$ in the state diagram of FMFA A. Hence, by Equalities 2, 3 and Definition 5,

$$M(A) = \bigcup_{\substack{i \in I_\gamma \\ q_f \in Q \\ F(q_f)>0}} \left\{ \left\{ \frac{\gamma(W_i(q_f))}{c(W_i(q_f)) \otimes F(q_f)} \right\} \oplus \bigoplus_{j=i_1}^{i_k} \left\{ \frac{\gamma(C_j)}{c(C_j)} \right\}^{\oplus} \right\}$$

which completes the proof. □

Remark 1. Part 1) of the previous proof implies that a fuzzy multiset language M, which can be expressed in semilinear form and which satisfies the condition $M(\mathbf{0}_\Sigma) \in \{0,1\}$, is accepted by an FMFA $A = (Q,\Sigma,\delta,q_0,F)$ with $F(q) \in \{0,1\}$ for all $q \in Q$.

6. Fuzzy multiset finite automata in reduced form

The fuzzy multiset finite automaton is described in Definition 4 with use of the fuzzy set of (final) states and the fuzzy transition relation. For later extending the corresponding theory, it will be advantageous to use reduced forms of FMFAs which contain the only fuzzy component and all other components are crisp (i.e., non-fuzzy). We can remind [36] as a seminal paper presenting such a reduced form for fuzzy finite automata. Reduced forms concerning some fuzzy multiset automata can be find for example in [37] and [38].

Definition 9. An FMFA $A = (Q,\Sigma,\delta,q_0,F)$ is called a *fuzzy multiset finite automaton with crisp transition relation* or *with crisp set of final states* if it satisfies the condition $\delta : Q \times \Sigma \times Q \to \{0,1\}$ or the condition $F : Q \to \{0,1\}$, respectively.

Note that Definition 9 can be adapted to FMFAs with extended transition relation as well. The next lemma can be easily obtained as a simple transformation of Theorem 3. It will be used in proof of Theorem 11.

Lemma 1. *Any fuzzy multiset language is accepted by an FMFA with crisp transition relation if and only if it is accepted by an FMFA with extended crisp transition relation.*

Theorem 10. *A fuzzy multiset language M over an alphabet Σ such that $M(\mathbf{0}_\Sigma) \in \{0,1\}$ is accepted by an FMFA if and only if it is accepted by an FMFA with crisp set of final states.*

Proof. Since every FMFA $A = (Q,\Sigma,\delta,q_0,F)$ with crisp set of final states is FMFA satisfying the condition $M(A)(\mathbf{0}_\Sigma) \in \{0,1\}$, it suffices to verify the reverse statement.

Let A be an FMFA and M be the fuzzy multiset language which is accepted by A and which satisfies the condition $M(A)(\mathbf{0}_\Sigma) \in \{0,1\}$. By Theorem 9, M can be expressed in semilinear form. Consequently, by Remark 1, there is an FMFA with crisp set of final states which accepts M. □

Remark 2. Note that the condition $M(A)(\mathbf{0}_\Sigma) \in \{0,1\}$ cannot be omitted in Theorem 10 because a fuzzy multiset language M such that $M(\mathbf{0}_\Sigma) = a \notin \{0,1\}$ is not accepted by any FMFA with crisp set of final states.

In the next theorem concerning FMFAs with crisp transition relation, a confinement of the structure of truth values is used. We remind that local finiteness of monoid $\langle L,\otimes,1\rangle$ means that each finite subset of L generates a finite submonoid.

Theorem 11. *Assuming locally finite monoid $\langle L,\otimes,1\rangle$, groups consisting of the following automata are equivalent:*

- *FMFAs,*
- *FMFAs with crisp transition relation.*

Proof. Since every FMFA with crisp transition relation is FMFA, it suffices to prove that for any FMFA, there is an equivalent FMFA with crisp transition relation.

Every fuzzy multiset language M_A accepted by an FMFA can be expressed in semilinear form by Theorem 9. By the definition of a fuzzy multiset language in semilinear form, M_A can be expressed as a union of languages in linear form. Since there is an obvious construction of an FMFA with crisp transition relation that accepts a union of fuzzy multiset languages accepted by FMFAs with crisp transition relation (see Theorem 6), it suffices to construct an FMFA with crisp transition relation that accepts a fuzzy multiset language M in linear form.

Let Σ be an alphabet and $M = \left\{ \left\{ \frac{\gamma_0}{c_0} \right\} \oplus \bigoplus_{j=1}^{m} \left\{ \frac{\gamma_j}{c_j} \right\}^{\oplus} \right\}$ for some $m \in \mathbf{N}$, $\gamma_0,\dots,\gamma_m \in \Sigma^\oplus$, $c_0,\dots,c_m \in L$.

Denote $I = \{l_M \mid M(\alpha) = l_M \text{ for some } \alpha \in \Sigma^{\oplus}\}$ — note that I is finite because of the assumption of locally finite monoid $\langle L, \otimes, 1\rangle$. Let $I = \{l_1, \dots, l_K\}$ with $K \geq 1$.

Clearly, for each $i \in \{1, \dots, K\}$, there are $i_1, \dots, i_m \in \mathbf{N}$ such that $l_i = c_0 \otimes c_1^{i_1} \otimes \cdots \otimes c_m^{i_m}$. Put $U = \max_{1\leq i\leq K}\{i_1, \dots, i_m\}$.

Consider an FMFA $B = (Q, \Sigma, \delta, \widetilde{q_0}, F)$ with extended transition relation where $Q = \{\widetilde{q_0}\} \cup \{q_{l_i} \mid 1 \leq i \leq K\}$ with $\widetilde{q_0} \neq q_{l_i}$ for all $i \in \{1, \dots, K\}$, F is defined by

$$F(q) = \begin{cases} c_0 & \text{if } q = \widetilde{q_0} \text{ and } \gamma_0 = \mathbf{0}_\Sigma, \\ l_i & \text{if } q = q_{l_i}, \\ 0 & \text{otherwise.} \end{cases}$$

and δ is defined by

$$\delta(q_i, \alpha, q_j) = \begin{cases} 1 & \text{if } \quad q_i = \widetilde{q_0},\ \alpha = \gamma_0 \text{ and } q_j = q_{c_0} \\ & \qquad \text{or} \\ & \quad q_i = q_{l_i},\ q_j = q_{l_j} \text{ and there are } k_1, \dots, k_m \in \{0, \dots, U\} \text{ such that } \alpha = \gamma_1^{k_1} \oplus \cdots \oplus \gamma_m^{k_m} \text{ and} \\ & \quad l_j = l_i \otimes c_1^{k_1} \otimes \cdots \otimes c_m^{k_m}, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, $\delta(q_i, \alpha, q_j) > 0$ for finitely many $\alpha \in \Sigma^{\oplus}$ which means that A is correctly defined FMFA with extended transition relation.

Verification, that $M = M(B)$:

1) Let $\alpha \in \Sigma^{\oplus}$ be such that $M(\alpha) < M(B)(\alpha)$. Since $M(B)(\alpha) > 0$, there are $q_f \in Q$, $J = \{j_1, \dots, j_n\} \subseteq \{1, \dots, m\}$, $k_1, \dots, k_n \in \mathbf{N}$ such that $\alpha = \gamma_0 \oplus \gamma_{j_1}^{k_1} \oplus \cdots \oplus \gamma_{j_n}^{k_n}$ and $M(B)(\alpha) = \bigvee_{q\in Q} \left\{\delta^*(\widetilde{q_0}, \alpha, q) \otimes F(q)\right\} = F(q_f)$ with $F(q_f) = c_0 \otimes c_{j_1}^{k_1} \otimes \cdots \otimes c_{j_n}^{k_n}$. Realizing the fact that $c_0 \otimes c_{j_1}^{k_1} \otimes \cdots \otimes c_{j_n}^{k_n} \leq M(\gamma_0 \oplus \gamma_{j_1}^{k_1} \oplus \cdots \oplus \gamma_{j_n}^{k_n}) = M(\alpha)$, we obtain contradiction with the assumed inequality $M(\alpha) < M(B)(\alpha)$.

2) Let $\beta \in \Sigma^{\oplus}$ be such that $M(B)(\beta) < M(\beta)$. Considerations similar to Part 1) of this proof lead again to contradiction.

Consequently, $M = M(B)$. With regard to Lemma 1, we conclude that Theorem 11 holds true. □

7. Conclusion

Fuzzy multiset regular languages (defined as languages generated by fuzzy multiset regular grammars) can also be characterized by means of fuzzy multiset finite automata, fuzzy multiset regular expressions, and as fuzzy multiset languages which can be expressed in a (newly described) fuzzy semilinear form. Fuzzy multiset finite automata have the same computational power as those with crisp set of final states (regarding a small requirement to membership value of the empty multiset) and also (assuming local finiteness of the structure of truth values) as those with crisp transition relation. Moreover, frequently used erroneous definitions of either nondeterministic and deterministic fuzzy multiset finite automata are mentioned to prevent their further repetition.

Hence, knowledge of the erroneous definitions should lead to thorough inspection of all papers that used them. Consequently, some of their results will need revision.

Generally, we can outline two basic directions for further research. One of them aims at more general fuzzy multiset languages, e.g., those accepted by deterministic and nondeterministic versions of fuzzy pushdown automata, fuzzy multiset linear bounded automata, and fuzzy multiset Turing machines. The other deals with various properties of fuzzy multiset regular languages or with solving various problems connected with fuzzy multiset finite automata like (approximate) equivalence or (approximate) minimization problems, etc.

Two final notes relate to the algebraic properties of the studied languages:

1. Myhill-Nerode theorem is a well-known characterization theorem for regular languages. However, it cannot be transformed to (fuzzy) multiset regular languages as can be easily checked with help of the multiset regular language $M = \{\langle a\rangle^n \oplus \langle b\rangle^n \mid n \geq 1\}$.
2. The investigation of closure properties of fuzzy multiset regular languages will be provided in another paper.

CRediT authorship contribution statement

Pavel Martinek: Writing - review & editing, Writing - original draft, Investigation, Formal analysis, Conceptualization.

Data availability

No data was used for the research described in the article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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