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Developing the concept of task substitution and transformation by defining own equivalences

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Abstract
The presented article is dedicated to a new way of teaching substitution in algebra. In order to effectively master the subject matter, it is necessary for students to perceive the equal sign equivalently, to learn to manipulate expressions as objects, and to perceive and use transformations based on defining their own equivalences. According to the results of several researches, these changes do not occur automatically, and the neglect of their development leads to students' insufficient adoption of substitution. The submitted contribution presents a new way of teaching substitution, the stages of which support the gradual development of the necessary competences of students, so that substitution becomes part of their computing apparatus. The effectiveness of the mentioned method of teaching substitution was also verified experimentally. By conducting a pedagogical experiment, it was confirmed that the application of the substitution teaching method developed by us led to more frequent use of substitution by students from the experimental group (47 students) compared to students from the control group (82 students) who learned substitution in the usual way. It emerged from the interview with experimental group students that they considered the proposed method suitable and that it encouraged them to learn substitution in depth.

Keywords Equivalences · Manipulation · Substitution · Transformation · Teaching mathematics

1 Introduction

Mathematics has a prominent place in the educational process because we encounter it and its applications quite often in everyday life (English & Gainsburg, 2015). The central topic of school mathematics is algebra (Brown et al., 2014). One of the reasons for emphasizing algebra in school mathematics is the fact that algebraic reasoning is considered an effective means of developing children's thinking (Blanton et al., 2015; Brizuela & Schliemann, 2004; Kaput et al., 2008). The basis of algebraic thinking can be considered the ability to generalize and at the same time to recognize and create connections between individual mathematical knowledge (Cai & Knuth, 2011). According to the findings of Kirshner and Awtry (2004), students' success in algebra correlates with how the individual student managed the transition from arithmetic to algebraic thinking.

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In this process, it is important that students overcome the obstacles associated with the transition to symbolic algebra (Filloy et al., 2008), because within the teaching of algebra, a large space is devoted to solving equations and inequalities. In both cases, the ability to manipulate algebraic expressions is important (Pedersen, 2015). When manipulating algebraic expressions, one is transformed into an expression equivalent to it. The equivalence of two expressions is the basis for substitution use, which often occurs when solving algebraic problems, but also problems from higher mathematics. Several research studies report students' problems in the application of substitution in task solving. de Lima and Tall (2008) reported that high school students tend to use memorized rules (e.g., moving the unknown from one side of the equation to the other) rather than substitution solutions (e.g., substituting $m=0$, into the equation) when solving equations with an unknown on both sides of the equation $2m=4m$). Instead of looking for the value of the unknown that fits the equation, there is a general tendency to move the symbols toward the desired solution (de Lima & Tall, 2008). Filloy et al. (2010) found that when solving algebraic equations, some high school students could often replace the unknown with a number (e.g., $y = 2$), but could not replace the variables with the expression containing another variable (e.g., $y = 3x + 1$). The wide spectrum of substitution application and students' difficulties in solving tasks associated with substitution are the reason for the creation of appropriate methods of teaching substitution (Jones et al., 2012). Considering the mentioned problems, our goal was to develop and experimentally verify an innovative method of teaching substitution in selected parts of high school mathematics curriculum.

2 Theoretical basis

In mathematical terms, algebraic substitution involves (a) replacing a more complex expression with one variable and (b) replacing one variable with a more complex expression (Jupri et al., 2016). The very concept of substitution — replacing one representation with another — is based on the formal mathematical definition of equivalence. If the relation is reflexive, transitive, and symmetric, it is an equivalence relation (e.g., Fischer et al., 2019; Stewart & Tall, 2015).

The concept of mathematical equivalence, specifically the symbolic representation of equivalence relation using sign "=", is the accepted basis of algebraic thinking (Carpenter et al., 2003; National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA & CCSSO], 2010 & Stephens et al., 2021). Equivalence expressed by sign "=" is a relationship between two mathematical expressions, on the basis of which we can claim that the expressions have the same value, or that the expressions represent the same mathematical object (Kieran & Martínez-Hernández, 2022). According to several researches (e.g., Booth et al., 2014; Byrd et al., 2015; Fyfe et al., 2018; Fyfe & Brown, 2020; Knuth et al., 2006; Matthews et al., 2012), the ability to interpret sign "=" is a relationally relevant predictor of the student's future success in algebra. The relational conception of the equal sign means that sign "=" indicates the "sameness" of two objects or expressions (Carpenter et al., 2003; Kieran, 1981; Knuth et al., 2005; McNeil et al., 2011). The relational perception of sign "=" allows us to interpret the symbolic notation $4x^2-1=(2x-1)(2x+1)$: the expressions $4x^2-1$ and $(2x-1)(2x+1)$ have the same value for the permissible values of x. Although the relational conception of "=" sign is an important basis for understanding mathematical equivalence (e.g., Matthews & Fuchs, 2020; Simsek et al., 2019), Jones and Pratt (2012) argue that a full relational conception of the equal sign involves more than understanding "sameness".

They propose that a complete relational view of equal sign includes a substitution component in addition to “sameness”. The substitution concept of “=” sign is based on the transitive and symmetric properties of equivalence (Simsek et al., 2019), which allow equivalent terms to be substituted for each other in mathematical equations. This concept allows equal sign to be interpreted as the equivalence of different-looking mathematical symbolic notations. Their equivalence (expressions have the same value) consequently enables their mutual interchangeability. The substitution concept makes it possible to replace number 14 in the expression $14+8$ with the expression $12+2$, because $14=12+2$ (Jones & Pratt, 2012), which helps simplify the calculation. In the case of the algebraic notation $4x^2-1=(2x-1)(2x+1)$, the substitution concept of “=” sign means that the expression $4x^2-1$ can be replaced by the expression $(2x-1)(2x+1)$ and vice versa. The original research by Jones et al. (2012) was not clear on the issue of the developmental arrangement of the relational and substitutional concept of equal sign. Later research (Simsek et al., 2019) suggests that sameness develops in students before substitution conception. These findings are consistent with the assertion of Donovan et al. (2022), who state that the substitution concept of equal sign follows logically from the sameness concept. At the same time, they point to the fact that the sameness concept of “=” sign is the necessary precursor to the creation of correct substitution concept. This opinion is also supported by the results of Lee and Pang (2021), who found that students who had a substitution concept of equal sign without a sameness concept learned the substitution rules but applied them in a way that was not consistent with the sameness concept. The proximity of the mentioned concepts of the “=” sign will be manifested in tasks that explicitly require students to rewrite the equality in a way that demonstrates its truth value. When solving such tasks, students may become sensitive to value-preserving substitutions. It is possible that in such arithmetic tasks, the differences between relational and substitutional conceptions of the equal sign are lost (Kieran & Martínez-Hernández, 2022). When solving algebraic problems, the sameness concept of “=” sign is not sufficient, because in algebra it is important for students to distinguish between equality (the same value) and sameness concept: to be equal does not mean to be the same (Asquith et al., 2007; Behr et al., 1980). In order for the student to be able to effectively apply the equivalence of equal sign, they need to correctly interpret algebraic notations. The equivalence of two expressions, which is symbolically expressed by “=” sign, should be interpreted as follows: the given two expressions have the same value and are interchangeable. Such an interpretation of equivalence is sufficient because it is essential for the understanding of the substitution concept that the mathematical definition of equivalence does not make any demands on properties a and b beyond how they are related through the equivalence relation. Formally, $a=b$ does not mean that a and b are “the same”, only that they are related by a given equivalence relation (Jones et al., 2012); i.e., they are interchangeable due to the same value.

Substitutions use not only symmetric, but also transitive properties of equivalence (Simsek et al., 2019). From transitivity point of view, two different-looking algebraic notations are equivalent if we can transform one into the other (Sfard, 2008). For a correct understanding of transitivity of equivalence and the associated transformations of expressions, the so-called structural sense is needed, that is, the ability to “use equivalent structures of expression flexibly and creatively” (Linchevski & Livneh, 1999, p. 191). In school algebra, structural sense means a set of abilities, such as: see part of an expression as a unit; split an expression into meaningful subexpressions; recognize which manipulation is possible and useful to perform; and choose appropriate manipulations that make the best use of the structure of the expression (Hoch & Dreyfus, 2005; Molina et al., 2017). Structural sense can be built by encouraging students to think of algebraic expressions as objects rather than just arithmetic procedures to be performed. In this reasoning, they should realize that the different representations of the same algebraic expression that they create while manipulating it have the same value and are interchangeable, i.e., are equivalent (Banerjee & Subramaniam, 2012). Students who have adopted a structural view of algebraic expressions can recognize the equivalence of the expressions $4x^2-1$ and $(2x-1)(2x+1)$ without having to perform individual calculations (Carpenter et al., 2003; Sfard, 1991), because the written structure $4x^2-1=(2x-1)(2x+1)$ corresponds to the "formulaic equivalence" $a^2-b^2=(a-b)(a+b)$.

In solving algebraic equations, students encounter one more type of symbolic notation, which also represents equivalence. Such an equivalence is, for example, the notation $4x-2=y$. The stated equivalence, however, is not the result of manipulations; i.e., the expression $4x-2$ cannot be transformed into the expression y. When solving algebraic equations, the equivalence $4x-2=y$ is created by the problem solver and can be perceived as information. The solver informs that they defined a new ("custom") equivalence that they will use to simplify or transform the task. With this type of equivalence, the student must realize that in the notation $4x-2=y$, it is necessary to distinguish between the variable y, which represents a number, i.e., the value of the expression $4x-2$, and the expression $4x-2$ itself, which represents one object (Dreyfus & Thompson, 1985). Such "own" equivalences are the basis for the use of substitution in solving equations, but also problems from higher mathematics. Although school algebra includes transformational aspects, too much focus on following rules when manipulating symbols will cause a lack of conceptual understanding of transformational activities (Kieran, 2007). Learners see expressions mainly as sets of symbols to be manipulated according to transformational rules, which may be arbitrary or well-reasoned by being based on a robust understanding of the structure of the expression (Kieran, 2004; Papadopoulos & Gunnarsson, 2020). Focusing on manipulations of expressions leads students to favor process strategies in solving equations.

For example, when solving the equation $(x-1)^2-1=2(x-1)+4$, the process strategy manifests itself in the fact that the student transforms the equation into the form $x^2-4x-2=0$ (Jupri & Sispiyati, 2017). If the student uses the above-mentioned sense of the structure of the expression, they will recognize the "repeating" subexpression x-1 in the given equation and will use the opportunity to simplify the solution of the given equation by defining the equivalence $x-1=y$. In this case, the student will use the structure perception strategy to solve the equation. A student can use this strategy if they (1) recognize a familiar structure in its simplest form; (2) can treat a compound expression as a single entity and recognize a familiar structure in a more complex form; and (3) choose appropriate manipulations for the best use of the structure (Novotná & Hoch, 2008). Using the strategy of perceiving structure of the given equation allows the students to effectively connect their various knowledge of mathematics by using appropriate substitution — defining their own equivalences, they can transform an unknown task into a task that they are already able to solve (Gonda et al., 2022). The ability to transform a problem, or part of it, is the basis of success in higher mathematics and in problem solving in general (Feikes & Schwingendorf, 2008). Research shows that the purposeful inclusion of transformational activities in teaching mathematics has a positive effect on the development of students' algebraic thinking (Agoestanto & Sukestiyarno, 2019; Ayalon & Even, 2015; Pedersen, 2015). It follows from the overview that the substitution concept is a complex concept, and it is important to research the way of teaching substitution as an effective method of solving a wide range of tasks.

3 The suggested way of teaching the substitution method

As already mentioned, an essential element of using substitution when solving equations is recognition of the repeated expression $V(x)$ in the given equation, which allows the solver to define their own equivalence in the form $V(x) = y$. In mathematics, equivalence is expressed by the sign "=". However, a lot of students hold an operational conception of equal sign, interpreting the sign as "total" or "answer" (e.g., Knuth et al., 2006; McNeil, 2008; Rittle-Johnson et al., 2011). The operational concept of "=" sign can cause students to reject non-standard equations, such as $x - 1 = y$. In the same way, students reject equality $7 = 2 + 5$ because it is "backwards" (Stephens et al., 2022). Thus, operational concepts can cause students to perceive the rule $x^n.\ x^m = x^{n+m}$ operationally; i.e., x^{n+m} is the result of the product $x^n.\ x^m$ and not as the equivalence of the expressions $x^n.\ x^m$ and x^{n+m}; i.e., terms have the same value and are interchangeable. Therefore, we recommend dividing the teaching of substitution use in solving equations into three phases. In the first phase, we focus on developing the perception of "=" sign as the equivalence of two expressions. In the second phase, we focus on the strategy of perceiving the structure of the expression, and in the final third phase, we focus on the development of transformational skills. We will now describe the three phases of the proposed method of teaching the use of substitution in solving equations in more detail.

3.1 Development of the equivalence concept of the equality of algebraic expressions

This phase can be implemented as part of curriculum by modifying algebraic expressions. A common task is e.g., factorize the expression $x^2 - 1$. The result is the notation $x^2 - 1 = (x - 1)(x + 1)$. Substituting different numbers for variable x, students can convince themselves that the expression $x^2 - 1$ has the same value as the expression $(x - 1)(x + 1)$. Now we ask them which symbol in the notation of our result expresses this fact. Finally, we emphasize that we present equality sign "=" as a symbol that expresses the equivalence of two expressions, i.e., given two different looking expressions have the same value and are interchangeable. We then present the students with numerous examples from life, where each time we use sign "=" to express the equivalence of different things based on their equivalence. For example, in the case of an exchange rate ticket, it is an equivalence relation between different currencies, which enables two-way exchange of money from one currency to another. Similarly, various mathematical or physical formulas express equivalence between quantities. Mathematical formulas, which are used when editing expressions, are also interpreted in the same way. For example, the formula $x^n.\ x^m = x^{n+m}$ due to the presence of sign "=" is the equivalence of two expressions. Their equivalence allows them to be interchanged, and thus this formula is a general guide as to how and when it is possible to replace the product of two powers with one. The given formula allows us to replace e.g., the expression $y^2 \bullet y^5$ by the expression y^7, but also y^7 by the expression $y^2 \bullet y^5$. At the same time, we can effectively justify why $3^2 \bullet 3^5 \neq 9^7$. In this way, we can point out the fact that modification of algebraic expressions does not consist only in calculations (as in modification of numerical expressions), but in the mutual substitution of equivalent, while the basic equivalences are expressed in formulas. We interpreted the call to "modify the expression" to the students as:

a. Discover one of the sides of the "formulaic substitution" and exchange for the other side;
b. Create one of the sides of the "formulaic substitution" and exchange it for the other side.

In the simple tasks where it was enough to discover so-called formulaic substitution, the students did well with the standard manipulation of expressions. Gradually, we mainly included tasks where it was necessary to create one of the sides of "pattern substitution" and then replace it with its so-called formulaic equivalent. The purpose of these tasks is to support the development of targeted manipulations with expressions. An essential element of this phase of teaching the substitution method was the frequent use of examples where it was necessary to use formulaic substitutions "both ways".

3.2 Perception of the structure of algebraic expressions

In the previous phase, students worked with equivalent expressions, the equivalence of which was expressed by "=" sign. The equivalence of the expressions allowed students to use substitution when manipulating the expressions. When solving algebraic equations, students encountered another form of equivalence, which was not predicted by mathematical formulas, but was created (defined) by the solver of the given equation. Whether it is appropriate to define own equivalence depends on the structure of the given equation. For students to consider this possibility, it is necessary that, in addition to operational strategies, they should be able to use the strategy of perceiving the structure of the given equation (Novotná & Hoch, 2008). Defining own equivalences is an intervention in memorized methods of calculation, which are often resistant to changes in students (Knuth et al., 2006). In our teaching method, we present the use of substitution as an alternative option for solving equations. In doing so, it is important to point out the fact that, based on the structure of the given equation, it is sometimes appropriate to define one's own equivalence, which can make the process of solving the given equation more efficient. Therefore, in this phase, we recommend including tasks of the following type:

$$\text{On the set } \boldsymbol{R} \text{ solve the equation } \left(\frac{x-3}{x+2}-5\right)\left(\frac{x-3}{x+2}+3\right)=9.$$

It would be appropriate for the students to solve the given equation first on their own in the already learned way. Usually, students proceed by creating one fraction in both brackets, which they then multiply and after several adjustments obtain the quadratic equation $x^2+4x+3=0-$process strategy. By solving it, we obtain a set of solutions of the given equation $\boldsymbol{K}=\{-3;-1\}$.

Subsequently, through a discussion form with students, the alternative solution to the given equation is presented, which is based on the presence of recurring (sub)expression $\frac{x-3}{x+2}$. As part of the discussion, we are moving towards the fact that the given expression can be considered as one object that can be manipulated. We demonstrate this fact as follows:

$$\left(\frac{x-3}{x+2}-5\right)\left(\frac{x-3}{x+2}+3\right)=\left(\frac{x-3}{x+2}\right)^2-5\left(\frac{x-3}{x+2}\right)+3\left(\frac{x-3}{x+2}\right)-15=\left(\frac{x-3}{x+2}\right)^2-2\left(\frac{x-3}{x+2}\right)-15.$$

The next step on the way to using substitution in solving equations is another consideration of the expression — the object $\frac{x-3}{x+2}$. This expression takes on different values, depending on the value of the variable x. It is often necessary to manipulate such an object (as a whole) while solving a task. Therefore, it is effective to replace this expression (object) with a symbol that represents its value. So, at the appropriate moment of solving the task, we use this option by writing it in the form

$$\frac{x-3}{x+2}=y.$$

This notation is information that in a certain section of the problem solving, the expression $\frac{x-3}{x+2}$ will be replaced by symbol — variable y, which represents its permissible values. In this way, we will let the students know that during the solution of the equations it is also possible to define our own equivalences in order to make the solution of the equation more efficient. Defining one's own equivalence when solving equations is called introducing substitution. Introducing substitution $\frac{x-3}{x+2}=y$ allows us to pass to the solution of the quadratic equation

$$y^2-2y-24=0.$$

Its roots are numbers $y_1=6$ and $y_2=-4$. If we know the permissible values of variable y, we also know the permissible values of the expression $\frac{x-3}{x+2}$. By solving the equations

$$\frac{x-3}{x+2}=6 \text{ or } \frac{x-3}{x+2}=-4$$

we calculate the corresponding values of unknown x, which are also the solution of the given equation: $\boldsymbol{K}=\{-3;-1\}$.

In the later part of this phase, tasks were also solved, where a "repeating expression" had to be created with suitable modifications. The goal of this phase was for the students to pay attention to the structure of the given equation and create a recurring expression by appropriate manipulations with the subexpressions, i.e., to use the structure perception strategy. For example, when solving the equation

$$\left(\frac{x-1}{x+1}-2\right)\left(\frac{x+1}{x-1}-1\right)=\frac{1-x}{x+1}$$

manipulations with sub-expressions with a similar structure were first implemented, the aim of which was to create the repeating expression $\frac{x-1}{x+1}$:

$$\frac{x+1}{x-1}=\frac{1}{\frac{x-1}{x+1}} \text{ and } \frac{1-x}{x+1}=-\frac{x-1}{x+1}$$

and subsequently the self-equivalence $\frac{x-1}{x+1}=y$ was defined.

3.3 Task transformation using substitution

Transformational skills, which Kieran (2007) classifies as core to school algebra, are linked to the use of substitution. In this phase, we expand the possibilities of using substitution to students. We use substitution not only to make solving an equation more efficient, but also to transform it into another type of equation. The utility of equation transformation can be presented in the following pair of examples. Students first solve the equation on the set R:

$$\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}+4\sqrt{x}-\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)=8. \tag{1}$$

After solving it, the same students solve the equation on the set R:

$$\left(a-\frac{1}{a}\right)\left(\frac{a+1}{a-1}+4a-\frac{a-1}{a+1}\right)=8. \tag{2}$$

First, their success in solving the mentioned equations is evaluated. In the discussion, students compare the difficulty of these equations. Then both equations are written on the board next to each other and a discussion takes place about the similar structure of both equations — the "only" difference is that the expression $\sqrt{x}$ in Eq. (1) was replaced by the expression a and thus Eq. (2) was created. Thus, by introducing the substitution $\sqrt{x}=a$, Eq. (1) was transformed into Eq. (2). This presents students with the possibility of using substitution to transform the assigned task into another task, the solution of which is easier for them. This approach is used in school mathematics primarily when solving irrational, logarithmic, exponential, and trigonometric equations, which are often transformed into a quadratic or linear equation. It is advisable to develop students' transformational skills by solving various equations with the same solution strategy: (1) adjust the equation to a suitable form, (2) transform it into a quadratic or linear equation by suitable substitution (or in another way). It is important for students to often encounter tasks whose solutions include modifications, enabling the transformation of the given task into a task that they already know how to solve, because this contributes to the development of their algebraic thinking (Ayalon & Even, 2015). One of the goals of this part of substitution teaching is the use of transformations to move from the so-called unsolvable task to solvable task. For example, a process strategy for solving the equation

$$(x+1)(x+2)(x+3)(x+4)=1$$

leads to the equation

$$x^4+10x^3+35x^2+50x+23=0,$$

which is unsolvable at the high school level. By analyzing the structure of the equation, it is possible to discover the rewriting of the equation into the form

$$(x+1)(x+4)(x+2)(x+3)=1$$

and after multiplying we get the equation

$$\left(x^2+5x+4\right)\left(x^2+5x+6\right)=1,$$

which contains a repeating expression, and by introducing substitution, we transform the problem into solving a quadratic equation.

An essential element of the above-described form of substitution teaching is the comparison of the already learned task-solving procedure with the solution procedure in which substitution was used. Since it is more challenging for the learner to memorize two algorithms simultaneously, comparison tends to suppress simple memorization and instead forces the learner to actively engage in understanding the underlying structures of the examples presented (Kang & Pashler, 2012; Mitchell et al., 2008). Although this approach may lead to more mistakes at the beginning and slow down the learning process, it tends to improve long-term knowledge retention and transfer (Rohrer & Pashler, 2010; Rohrer & Taylor, 2007). At the same time, insight into the structure of the task solution procedure allows them to better understand the task transformation.

4 Methodology

Pedagogical research took place in selected secondary schools in Slovakia, with the consent of the school management. The goal of the research was to verify whether our suggested method of teaching substitution initiates students' use of the strategy of perceiving the structure of the given equation, i.e., considering using the option to define own equivalence and use it to transform the task. As part of the research, we set research questions:

RQ1: What change in the strategy of solving equations will be caused by our proposed method of teaching substitution?

RQ2: How will students react to an alternative way of solving equations for them?

The research sample consisted of first-year students at selected grammar schools in Slovakia aged 15 to 16. By random selection, we divided the students who voluntarily decided to participate in the research into two groups: experimental (EG) and control group (CG). The experimental group consisted of 47 students and the control group consisted of 82 students. Both groups were given a mathematical test (pre-test) before the start of the experiment, which was used to verify the equivalence of EG and CG. The pre-test was conducted at the beginning of the school year before the teaching equations. The content of the pre-test were five standard mathematical tasks, focused on modifying expressions and solving linear equations at the level of the elementary school curriculum. Based on the analysis of the pre-test results, which is presented in the section Analysis of the research results, we consider both groups CG and EG to be equal.

In the suggested way, which is described above, teaching substitution took place in the experimental group EG within the thematic units of modifying expressions, quadratic equations, irrational equations, exponential equations, and logarithmic equations. The experimental teaching was conducted over the course of 3 months, in the same number of hours as the teaching in CG. At the end of the experimental teaching, unstructured (open) interview was conducted with randomly selected students from EG (14 students). The interview was aimed at finding out whether the students in EG registered a difference compared to the way of teaching mathematics that they were used to and what was the essential difference and what benefit substitution represents for them, as an alternative way of solving equations.

The teachers in the experimental group were previously trained to teach substitution in the way we proposed. One of the authors functioned as a consultant for EG teachers in the framework of their preparation for individual lesson teaching. In control group, some teachers from selected schools taught substitution in a standard way — as a way of solving (algorithm) some equations that correspond to the scheme

$$a(V(x))^{2n} + b(V(x))^{n} + c = 0. \tag{3}$$

The students of control group learned that if the entered equation has the form of Eq. (3), then we use the substitution in form $y = V(x)$. Subsequently, students in CG solved equations where they used substitution if the given equation contained "repeating expression", even if it did not have the equation form (3). Such two types of equations (often with an instruction in the assignment for students to solve the given equation using suitable substitution) are in collections that were used by both teachers and students during classes in CG.

In both groups, mostly the same tasks were solved, but in EG, emphasis was placed on the development of the above-described concepts and skills related to the use of substitution. In CG, teaching was focused on learning a new method (substitution) for solving a certain type of equation.

According to Freudenthal (1986), a student who has gone through the learning process cannot generally be assumed to have mastered a new solution method sufficiently to be able to reconstruct it. If the student can reconstruct a new way of solving tasks even after some time has passed, it can be said that the student has mastered it with understanding (Fan & Bokhove, 2014). Therefore, we administered the post-test to both groups five months after the end of the experimental teaching. As a criterion for the effectiveness of the proposed method of teaching substitution, we chose to compare the frequency of using substitution when solving the post-test. Based on the above, we expressed the following research hypothesis.

H: Students who undergo the innovative method of teaching substitution use substitution more often when solving mathematical problems.

After 5 months from the end of the experimental teaching (planimetry and structural geometry were taught at that time), we once again gave the students of both groups (EG and CG) a mathematical test (post-test) focused on solving equations, while we again observed for each student, whether they used substitution when solving the assigned task. All the tasks were selected from the collection of tasks used by teachers in both groups during the teaching of solving individual types of equations. We assumed that students in both groups would solve tasks 1 and 2 mainly by using substitution. We expected differences in the frequency of using substitution when solving tasks 3 to 5. Solving tasks 4 and 5 without using substitution leads to equations, the solution of which is beyond the scope of the secondary school curriculum. Therefore, we were interested in whether students, after reaching an unsolvable equation for them, would consider the possibility of transforming the task by defining a suitable equivalence.

We also chose the time gap (5 months) from the teaching with the innovative method to verify whether the new substitution teaching method affects the students' way of solving tasks even after a long time since its completion. The tasks in the post-test were also the same for both groups (CG and EG) and are listed in the Appendix.

We used selected statistical methods to analyze the results that the students achieved both in the pre-test and in the post-test. Since the assumption of a normal distribution of errors in pre-tests and post-tests was not fulfilled, we used a non-parametric method — the two-sample Wilcoxon signed rank test — to compare the results achieved by CG and EG students. The Wilcoxon two-sample test is used to evaluate the statistical significance of the differences between two independent sets as a non-parametric alternative to the parametric *t*-test in cases where the assumptions of using parametric tests are not met.

The tested hypothesis for the comparison of the results achieved by CG and EG students in both the pre-test and the post-test was the following null hypothesis H_0: There **is no** statistically significant difference between the two groups of students (control and experimental) regarding the results achieved in pre-test (post-test). We tested the null hypothesis against the alternative hypothesis H_1, that there **is** a statistically significant difference between the two groups of students (control and experimental) with regard to the results achieved in the pre-test (post-test).

We implemented the Wilcoxon two-sample test using the STATISTICA 10 program. We evaluated the test results based on the p value, which is the probability of the error we made when we rejected the tested hypothesis.

5 Analysis of pre-test and post-test results

5.1 Analysis of pre-test results

As stated in the methodological section, the pre-test tasks were assigned from the curriculum, which the students covered in the classic way. We do not list their assignment here, only in Fig. 1 we graphically show the success of both groups.

Using the Wilcoxon two-sample test in the STATISTICA program (Markechová et al. 2011), we obtained the following results: The value of the test statistic was $Z = 0.30$ and the probability value was $p = 0.76$. Since the *p-value* was greater than 0.05, we could not reject the null-hypothesis. That means that there **was no** statistically significant difference between CG and EG students in the results of the pre-test.

5.2 Analysis of post-test results

As part of the post-test, we monitored the method (strategy) used by the students in solving the assigned tasks. When analyzing the completed tasks of individual students, we recorded whether or not substitution was used.

Figure 2 shows how often students of both groups used substitution when solving individual tasks in the post-test.

If we compared the fact of how often CG and EG students used substitutions when solving individual tasks of the post-test (Fig. 2), we could conclude that when solving the first task, we saw that EG students used substitution almost as often as CG students. This is because it is the most basic type of problem in school mathematics, the solution of which uses substitution. However, in the other tasks of the test, a significant difference could be seen in how often students used substitutions when solving the given task — in favor of EG. In CG, the process strategy prevailed in the solution. Students tried to transform the entered equation into linear or quadratic equation by manipulating the expressions. In task 3, some students in CG defined the equivalence $y = \frac{x-1}{x+1}$ and then solved the equation

$$(y-2)(y-1) = \frac{1-x}{x+1}.$$

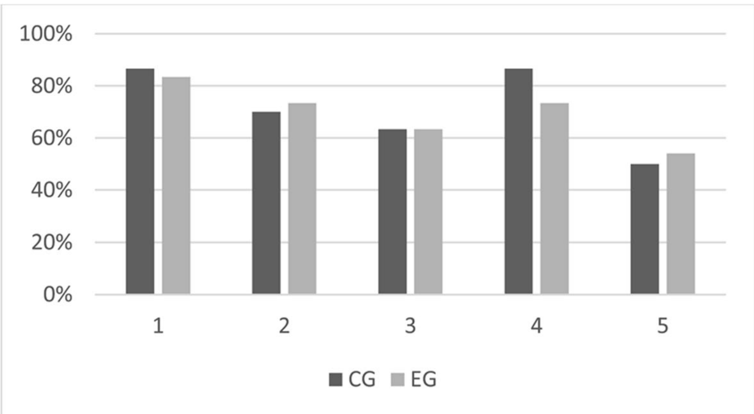

Fig. 1 Pre-test results. Source: self-made

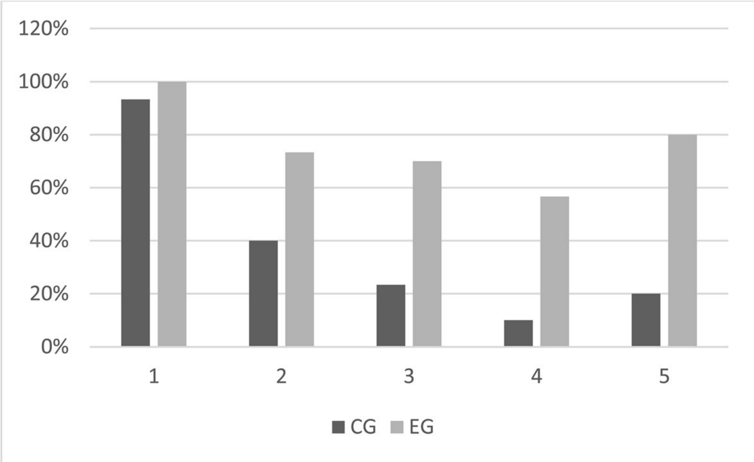

Fig. 2 Results of the post-test (frequency of use of substitution). *Source*: self-made

We also considered these cases to be the use of substitution, even if they led to the solution of an equation with two unknowns. A surprising fact was that in task 5 in CG, the frequency of using substitution was higher than in task 4, although that task contained a repeating expression (x^2+4x) and in task 5 the repeating expression had to be created. The CG students were able to produce the repeating expression although they used process strategy. The process strategy was manifested in the fact that the CG students transformed the equation through manipulations into the form $\left(\sqrt[3]{x}\right)^2 + 1 = \sqrt[3]{x} + 3$ (Jupri & Sispiyati,

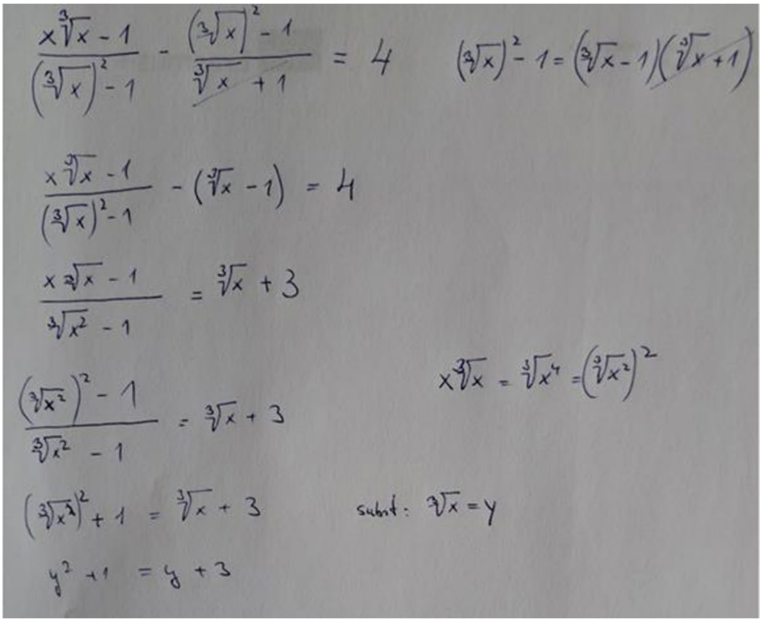

Fig. 3 Example of solving task5 in CG

2017) and used the substitution when the equation had the form (3). The procedure they used is shown in Fig. 3.

In EG, when solving problem 5, students mainly used the adjustment $x\sqrt[3]{x} = \sqrt[3]{x^4} = \left(\sqrt[3]{x}\right)^4$ and defined the equivalence $\sqrt[3]{x} = y$, which corresponded to the structure perception strategy.

To verify the statistical significance of the differences between CG and EG, we again used the Wilcoxon signed rank test, and we got the following results: the value of the test statistic $Z = 2.32$, the probability value $p = 0.020$. Since the calculated value $p < 0.05$, we rejected the null hypothesis. It means that EG students statistically significantly more often used substitution when solving mathematical problems than CG students. Based on the statistical analysis results, we concluded that the validity of the research hypothesis was confirmed; i.e., the proposed concept of teaching the substitution method had the potential to contribute to a more permanent acquisition of the use of substitution in solving equations.

5.3 Analysis of the results obtained by interviewing students

As part of the conducted interview, EG students expressed the opinion that the method of teaching substitution was different for them from the usual mathematics lessons. They saw the difference in the fact that a large part of the teaching was carried out in a way that led them to "think", to find the simplest possible solution and to finally evaluate the overall solution process. This difference was succinctly expressed by one student who said: "Until now I was used to counting in math class, and now I have a headache from thinking." During the interview, we found out what benefit the substitution teaching represented in the experiment. Students' answers could be summarized in the following areas:

1. *The need to analyze the assigned task* — the students expressed that they did not learn how to solve the task, but how to find an effective way to solve the task by thinking about it. They were aware that they did not have to automatically assign a learned calculation algorithm to the task assignment, but often the task can be transformed so that already acquired knowledge is used in the solution. Ivan gave an interesting answer: "I didn't realize until now how important it is to be able to solve quadratic equations well" or Janka: "I don't have to remember so many procedures."
2. *Atomization of the task* — during the interview, almost all students expressed a certain degree of surprise that the task does not have to be solved as a whole but can be solved by gradually solving sub-tasks. Jaroslav expressed it succinctly: "It is interesting that solving one task means solving several tasks."
3. *The interconnectedness of mathematical knowledge* — the students stated that they had to change their approach to solving the task. Most of them stated that they were aware that solving a task did not mean assigning the learning procedure to the task assignment. Rather, it was necessary to analyze the assignment and realize which knowledge was needed to solve the task. It was the atomization of the task that showed them that some sub-tasks from various parts of mathematics were often solved within one task. Thus, they would use multiple learned procedures. Student Eva expressed it very succinctly: "The assignment is not a barrier that limits the calculation possibilities."

Overall, the students expressed in the interview that they were satisfied with the method of teaching substitution and that it was beneficial for them (72%). Some students (19%)

expressed that they were more comfortable with the focus of mathematics lessons on practicing problem solving procedures. The rest of the students (9%) did not see the difference between the usual way of teaching and experimental teaching.

6 Discussion

Based on the statistical analysis of the post-test, a statistically significant difference was shown in the frequency of using substitution when solving tasks. In CG, substitution was taught as a method for solving a certain type of equation, one that corresponded to the first task of the post-test. When solving this task, up to 93% of CG students used substitution. For other tasks that did not correspond to the usual learned scheme for using substitution, only 23% of CG students used substitution on average. The majority of CG students did not use substitution even if they arrived at a complex or unsolvable equation through manipulations. Considering this finding, we concluded that substitution was not an alternative option for them to solve the given equation. At the same time, this finding implied that CG students did not examine the structure of the equation during the manipulations. Otherwise, they would have "noticed" that through the manipulations they had arrived at an equation structure that corresponded to the equation they had learned to solve by substitution. The analysis of the post-test in CG showed that the students memorized the procedures for solving equations, which is, according to Navarro-Ibarra et al. (2017), consequence of teaching mathematics based on the transfer of ready-made knowledge. CG students used a predominantly procedural strategy, which indicated a formal acquisition of substitution at the procedural level, which corresponded to instrumental understanding (Skemp, 1976). When solving the post-test tasks, they relied on learned procedures and rules (one of them was the rule of when to use substitution), which students often adopted without understanding (Fuson et al., 2005) and the necessary insight (Freudenthal, 1986). According to Kincheloe (2003), formal education based on the transfer of information from the teacher to the students does not contribute to the formation of concepts or to the improvement of the ability to solve tasks, which was confirmed in CG. In the experimental group, 100% of students used substitution in the "standard task" and in other tasks, an average of 70% of EG students used substitution. EG students in some cases solved the task first without using substitution. But as soon as they realized that the manipulations they performed led to a complicated solution, they went back to entering the equation and, by defining equivalence and subsequent substitution, simplified the solution of the equation. This confirmed that EG students adopted substitution as an alternative method of solving equations.

We think that the fact that in EG the tasks in the lesson were aimed at investigating the structure of the given equation and their solution in several ways (without substitution and with substitution) contributed to this. In this way, the development of the structure perception strategy was supported in the students.

Flexible use of a new way of solving tasks indicates its acquisition with understanding (Fan & Bokhove, 2014; Rittle-Johnson et al., 2015). Acquiring a computational skill with understanding corresponds to relational understanding (Skemp, 1976). Based on our experimental findings, we conclude that the proposed substitution teaching methodology has the potential to increase students' success in solving algebraic tasks.

It is also particularly important that the analysis of the interview with the EG students showed that the students also perceived the difference between experimental and regular teaching. As part of the experimental teaching, they were encouraged to analyze the task and think about the way to solve the assigned task, which are elements of deep learning (Marton & Säljö, 1976). Only some students (9%) expressed that they were more comfortable with the usual teaching method, which, according to research (e.g., Boaler, 2015; Fuson et al., 2005), was focused on calculations, and such a method of teaching mathematics is often preferred by students in secondary schools (Vermetten et al., 1999). The different way of learning substitution between CG and EG students and also the interview analysis indicated that the proposed substitution teaching methodology has the potential to change the way students learn. This clue could be the subject of further research into the implementation of the proposed methodology.

7 Study limitations

Some limitations of our study may have affected our results. Students participated in the pedagogical experiment on a voluntary basis. We did not examine the overall mathematical skills of the research participants before starting the experimental teaching. With the pre-test, we only verified whether CG and EG students have the same ability to solve simple equations. Potentially significant differences in the level of mathematical knowledge might have influenced their way of acquiring the substitution. Potentially, further studies could examine whether the level of students' mathematical skills affects the acquisition of alternative ways of solving problems. Another limiting factor could have been missing data on the preferred learning style of individual participants in the experiment. Our findings indicated that the proposed substitution teaching method has the potential to encourage students to adopt a deep learning style. It would be beneficial to verify whether teaching alternative ways of solving tasks is a suitable way for students of different learning styles and supports a deep learning style.

8 Conclusion

Through the implementation and subsequent analysis of the pedagogical experiment results, it was confirmed that the methodology proposed by us for teaching substitution method has the potential to increase the level of students' calculation skills and thereby increase their success in algebra. It turned out that the students of the experimental group used substitution more often to solve various equations, even after a period of time, and they evaluated the proposed method of teaching substitution positively. They identified the change in their perception of solving equations as the main benefit. They realized that it was not necessary to memorize a large number of calculation algorithms, but thanks to the understanding of substitution, it was possible to transform the task, which allowed to subsequently use the learned algorithm. It was confirmed that it was also useful to know that when solving a task, a student could go from solving a problem to mathematical investigation (Vidermanová et al., 2013), searching for a connection between the given problem and their knowledge. This way of thinking is based on "mathematical" freedom. It was useful for the student to know that during the solution they could perform any mathematically permissible operation that enabled them to achieve the partial goal set by them. Interviews with students showed that students had no problem with changing their learning style. They were comfortable with a teaching method that is based on understanding new knowledge.

The knowledge acquired in this way will be remembered by the student for a longer period of time and thus their success in solving tasks increases.

Appendix

Post-test

Task 1. Solve the equation on the set R $(x-5)^2-7(x-5)=44$.
Task 2. Solve the equation on the set R $\left(\frac{x-3}{x+2}-5\right)\left(\frac{x-3}{x+2}+3\right)=9$.
Task 3. Solve the equation on the set R $\left(\frac{x-1}{x+1}-2\right)\left(\frac{x-1}{x+1}-1\right)=\frac{1-x}{x+1}$.
Task 4. Solve the equation on the set R $\sqrt{x^2+4x+8}+\sqrt{x^2+4x+4}=\sqrt{2\left(x^2+4x+6\right)}$.
Task 5. Solve the equation on the set R $\frac{x.\sqrt[3]{x}-1}{\sqrt[3]{x^2}-1}-\frac{\sqrt[3]{x^2}-1}{\sqrt[3]{x}+1}=4$.

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Data availability Due to ethical concerns, supporting data cannot be made available.

Declarations

Conflict of interest The authors declare no competing interests.

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