Applications of the differential transform to second-order half-linear Euler equations

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ABSTRACT

Nonlinear differential equations are considered to be an important tool for describing a number of phenomena in engineering and the natural sciences, and their study is thus subject to contemporary research. The purpose of the paper is to show applications of the differential transform to second-order half-linear Euler equations with and without delay. The case of proportional delay is considered. Finding a numerical solution to an initial value problem is reduced to solving recurrence relations. The outputs of the recurrence relations are coefficients of the Taylor series of the solution. Validity of the presented algorithm is demonstrated on concrete examples of initial value problems. Numerical results are compared with solutions produced by Matlab function ''ddesd''.

Keywords: Half-linear Euler equation, differential transform, method of steps, differential equation with delay

1. Introduction

Half-linear Euler type equations have been studied extensively in terms of the qualitative properties. However, combination of research methodologies is one of the successful generators of new ideas, results, and insights. The aim of this paper is to complement the theoretical results about qualitative behaviour of solutions - asymptotic formulas or oscillatory properties - to half-linear Euler equations with and without delay with finding an approximate solution to the initial value problem numerically. Motivated by the current progress in research on the differential transform, the purpose of the paper is to investigate how the differential transform algorithm can be applied to half-linear Euler equations with a proportional delay and without delay.

The half-linear equation without delay can be achieved as a transformation of partial differential equations that contain the so-called p-Laplacian

$$\Delta_p u := \operatorname{div}(\|\nabla u\|^{p-2}\nabla u),$$

where for u(x) = u(x_1,... ,x_N), N ∈ ℕ, the symbol Vu stands for the Hamilton nabla operator and div represents the divergence operator. Origins of p-Laplacian are described, for example, in the paper [**1**]. Accordingly, the history of p-Laplacian is closely linked to applications in the filtration of fluids through porous media and nonlinear non-Newtonian fluid dynamics. Another application can be found, for example, in the paper [**2**]. The _p-Laplacian is used to model a nonhomogeneous diffusion to determine the height of a growing pile of noncohesive sand, where an ordinary differential equation arises in the limit case of “infinitely fast/slow” diffusion (see also [**3**]). More applications of nonlinear delayed differential equations are mentioned for example in [**4,5**].

The paper is organized as follows. In **Section 2**, we briefly summarize the results about qualitative properties of solutions to half-linear Euler equations with and without delay. **Section 3** is devoted to theory and results on the differential transform. **Sections 4** and **5** contain the main results of the paper. Numerical algorithms of the differential transform are adapted and applied to ordinary and delayed half-linear Euler equations. Concrete examples are illustrated by numerical results. **Section 6** concludes the paper with a summary and outlines possibilities for further research.

2. Half-linear differential equations

Half-linear differential equation of the second order is an equation of the form

$$(r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-1}\operatorname{sgn} x, \quad p > 1, \tag{1}$$

where r(t),c(t) are continuous functions and r(t) > 0. For p = 2, Eq. (**1**) reduces to the second-order linear Sturm-Liouville differential equation

$$(r(t)x')' + c(t)x = 0.$$

Also from this point of view, the study of the properties of its generalized form (**1**) is a natural direction of research.

The name ‘‘half-linear’’ has its origin in the fact that the space of solutions to (**1**) is homogeneous but not additive. The qualitative theory of half-linear differential equations has been studied extensively during the last decades. For the summary of the results up to 2005, we recommend the book [**6**]. More recent results can be found, for example, in [**7-13**] and references therein.

From the qualitative point of view, half-linear differential equations of the form (**1**) can be divided into two classes. A half-linear equation is either oscillatory, which means that every its nontrivial solution has infinitely many zeros that form a sequence tending to infinity, or nonoscillatory, which means that every solution has constant sign in a neighbourhood of infinity. e recall that an oscillatory solution and a non-oscillatory solution to (**1**) cannot exist at the same time, which is a direct consequence of Sturm separation theorem [**6**, p. 16, Theorem 1.2.3].

The subject of our interest in this paper is the second-order halflinear Euler equation

$$((x'(t))^{\alpha})' + \frac{\gamma}{t^{\alpha+1}} x^{\alpha}(t) = 0, \tag{2}$$

as well as the second-order half-linear Euler equation with a proportional delay

$$((x'(t))^{\alpha})' + \frac{\gamma}{t^{\alpha+1}} x^{\alpha}(\lambda t) = 0, \tag{3}$$

where a $>$ 0 is a quotient of two odd positive numbers, y ∈ (0, ∞) and λ ∈ (0,1).

Euler Eq. (**2**) and its generalized forms belong to the most studied half-linear differential equations, see, for example [**13-17**].

For r(t) = 1,c(t) = $\frac{r}{t^{a+1}}$ and p = α +1, Eq. (**1**) is reduced to the Euler Eq. (**2**). Half-linear Euler equation is conditionally oscillatory with the oscillation constant

$$\gamma_\alpha = \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1},$$

that is, (**2**) is oscillatory if $y > y_\alpha$ and non-oscillatory if $y < y_\alpha$.

If $\gamma = \gamma_\alpha$, Eq. (**2**) is also non-oscillatory and the two linearly independent solutions forming the solution space have the form

$$x_1(t) = t^{\frac{\alpha}{\alpha+1}},$$
$$x_2(t) \sim t^{\frac{\alpha}{\alpha+1}} \log^{\frac{2}{\alpha+1}}(t),$$

where ∼ means the asymptotic equivalence as t → ∞. It means that one of the solutions is known explicitly, whereas we only have an asymptotic formula for the second one. For details we refer to [**6**, Section 1.4.2].

Asymptotic formulas for the two linearly independent solutions are known also in the case $y < y_\alpha$ (see [**18**]). For i = 1, 2 these are of the form

$$x_1(t) = t^{\frac{\alpha}{\alpha+1}},$$
$$x_2(t) \sim t^{\frac{\alpha}{\alpha+1}} \log^{\frac{2}{\alpha+1}}(t),$$

where λ_i are the zeros of the equation

$$|\lambda|^{1+\frac{1}{\alpha}} - \lambda + \gamma = 0.$$

The Euler equation with a proportional delay (**3**) can be seen as a special case of the delayed half-linear equation

$$\left(r(t)\Phi(x'(t))\right)' + c(t)\Phi(x(\tau(t))) = 0, \tag{4}$$

where r(t), c(t), r(t) are continuous functions on $[t_0, \infty)$, $r(t) > 0$, and x is a delay function satisfying

$$\tau(t) \leq t, \quad \tau'(t) \geq 0, \quad \lim_{t\to\infty} \tau(t) = \infty.$$

In contrast to the non-delayed case (**1**), delayed half-linear equations may have oscillatory and non-oscillatory solutions simultaneously. A consequence of this fact is that techniques applicable to half-linear equations without delay (**1**) often cannot be applied to delayed half-linear equations. In particular, only Riccati type inequality is available instead of the Riccati equation. Moreover, to prove that a solution is oscillatory is easier than to prove that it is non-oscillatory.

If we choose $r(t) = 1, c(t) = \frac{\gamma}{t^{\alpha+1}}, p - 1 = \alpha$ and $\tau(t) = \lambda t$ in (**4**), we obtain the delayed Euler Eq. (**3**). Criteria providing conditions on γ under which (**3**) has only oscillatory solutions were studied, for example, in [**19,20**], the oscillation constant between oscillation and nonoscillation was derived in [**21**].

3. Differential transform

The Differential transform (DT) is a semi-analytical method based on Taylor's theorem. Its history dates back to 1970s to the work of G. E. Pukhov [**22**]. It has been shown that DT is convenient for solving a variety of initial value problems (IVPs), covering the range from ordinary to functional, partial and fractional differential equations [**23-26**]. In particular, results on the differential equations with proportional, constant and non-constant delays can be found in [**27-29**].

The differential transform of a real function $u(t)$ at a point $t_0 \in \mathbb{R}$ that is analytic in a neighbourhood of t_0 is

$$\mathcal{D}\{u(t)\}[t_0] = \{U(k)[t_0]\}_{k=0}^{\infty}. \tag{5}$$

Here $U(k)[t_0]$ is the fcth component of the differential transform of the function $u(t)$ at t_0, $k \in \mathbb{N}_0$, that is defined by

$$U(k)[t_0] = \frac{1}{k!}\left[\frac{d^k u(t)}{dt^k}\right]_{t=t_0}. \tag{6}$$

The inverse differential transform of $\{U(k)[t_0]\}_{k=0}^{\infty}$ is defined by

$$u(t) = \mathcal{D}^{-1}\left\{\{U(k)[t_0]\}_{k=0}^{\infty}\right\}[t_0] = \sum_{k=0}^{\infty} U(k)[t_0](t-t_0)^k. \tag{7}$$

In applications, the function $u(t)$ is approximated by the finite sum

$$u(t) = \sum_{k=0}^{N} U(k)[t_0](t-t_0)^k.$$

As we can see from (**7**), DT is related to the Taylor series. It means that the results about convergence of Taylor series may be used to decide on convergence of the DT algorithms. However, we refer to the particular paper [**30**] where the optimal general explicit a-priori error estimates are given.

The following results will be used in the application **Sections 4** and **5**.

Lemma 3.1. Assume that $u(t)$ is a real analytic function near t_0.

Then

$$U(0)[t_0] = u(t_0), U(1)[t_0] = u'(t_0), U(2)[t_0] = \frac{u''(t_0)}{2}, U(3)[t_0] = \frac{u'''(t_0)}{3!}, \dots . \tag{8}$$

The statement is directly implied by (**6**). The relationships (**8**) will be used when transforming initial conditions.

Lemma 3.2. Assume that $U(k)[t0]$ is the kth component of the differential transform of the real analytic function $u(t)$at t_0.

Then

$$D\{u'(t)\}[t_0] = \{(k+1)U(k+1)[t_0]\}_{k=0}^{\infty}. \tag{9}$$

Proof. Using (**7**), we can write

$$\begin{aligned} u'(t) = \frac{d}{dt}\sum_{k=0}^{\infty} U(k)[t_0](t-t_0)^k &= \sum_{k=1}^{\infty} kU(k)[t_0](t-t_0)^{k-1} \\ &= \sum_{k=0}^{\infty}(k+1)U(k+1)[t_0](t-t_0)^k. \quad \square \end{aligned}$$

Lemma 3.3 ([**29**]).

Assume that $F(k)[t_0]$ is the kth component of the differential transform of the function $f(t)$at t_0 and $r \in \mathbb{R}$.

$$\textit{If} \quad f(t) = t^r, \quad \textit{then} \quad F(k)[t_0] = \binom{r}{k} t_0^{r-k} \tag{10}$$

for all t such that $|t - t_0| < |t_0|$, where $\binom{r}{k} = \frac{r(r-1)\dots(r-k+1)}{k!} = \frac{(r)_k}{k!}$,

and $®_k$ represents the Pochhammer symbol.

Lemma 3.4 ([**28**]).

Assume that $F(fc), G(k)$ are the kth components of differential transforms of functions $f(t), g(t)$ at a point t_0. Differential transform of a product $f(t)g(t)$ at t_0 is

$$\mathcal{D}\{f(t)g(t)\}[t_0] = \left\{ \sum_{l=0}^{k} F(l)G(k-l) \right\}_{k=0}^{\infty}. \tag{11}$$

Lemma 3.5 ([**31**]).

Let g and f be real functions analytic near t_0 and $g(t_0)$, respectively, and let h be the composition $h(t) = (f_{\circ g})(t) = f(g(t))$. Denote $\mathcal{D}\{g(t)\}[t_0] = \{G(k)\}_{k=0}^{\infty}$, $\mathcal{D}\{f(t)\}[g(t_0)] = \{F(k)\}_{k=0}^{\infty}$

and $\mathcal{D}\{(f \circ g)(t)\}[t_0] = \{H(k)\}_{k=0}^{\infty}$

the differential transforms of functions g, f and h at t_0, $g(t_0)$ and t_0, respectively. Then the numbers $H(k)$ in the sequence $\{H(k)\}^{\infty}_{k=0}$ satisfy the relations $H(0) = F(0)$ and

$$H(k) = \sum_{l=1}^{k} F(l) \cdot \hat{B}_{k,l}\big(G(1), \ldots, G(k-l+1)\big) \;\; \textit{for } k \geq 1, \tag{12}$$

where $\hat{B}_{k,l}(\hat{x}_1, \ldots, \hat{x}_{k-l+1})$ are the partial ordinary Bell polynomials.

Lemma 3.6 ([**31**]).

The partial ordinary Bell polynomials $\hat{B}_{k,l}(\hat{x}_1, \ldots, \hat{x}_{k-l+1}),\ l = 1, 2, \ldots, k \geq l,$, satisfy the recurrence relation

$$\hat{B}_{k,l}(\hat{x}_1, \ldots, \hat{x}_{k-l+1}) = \sum_{i=1}^{k-l+1} \frac{i \cdot l}{k} \hat{x}_i \hat{B}_{k-i,l-1}(\hat{x}_1, \ldots, \hat{x}_{k-i-l+2}), \quad \text{where} \quad \hat{B}_{0,0} = 1 \textit{ and } \hat{B}_{k,0} = 0 \textit{ for } k \geq 1$$

Accordingly, the first few polynomials $\hat{B}_{k,l}(\hat{x}_1, \ldots)$ are:

$$\begin{array}{llll} \hat{B}_{0,0} = 1 & & & \\ \hat{B}_{1,0} = 0 & \hat{B}_{1,1} = \hat{x}_1 & & \\ \hat{B}_{2,0} = 0 & \hat{B}_{2,1} = \hat{x}_2 & \hat{B}_{2,2} = (\hat{x}_1)^2 & \\ \hat{B}_{3,0} = 0 & \hat{B}_{3,1} = \hat{x}_3 & \hat{B}_{3,2} = 2\hat{x}_1\hat{x}_2 & \hat{B}_{3,3} = (\hat{x}_1)^3. \end{array} \tag{13}$$

Lemma 3.7 ([**32**]).

Let $D\{f(t)\}[t_0] = \{F(k)[t_0]\}_{k=0}^{\infty}$ be the differential transform of the function $f(t)$ at t_0. Then the components $F(k)[t_1]$ of the differential transform $D\{f(t)\}[t_1] = \{F(k)[t_1]\}_{k=0}^{\infty}$ *of* $f(t)$ *at* $t_1 > t_0$ may be expressed as

$$F(k)[t_1] = \sum_{j=0}^{\infty} \binom{k+j}{j} (t_1 - t_0)^j F(k+j)[t_0], \quad k \geq 0. \tag{14}$$

4. Application of the differential transform to the Euler equation

Consider the initial value problem for the half-linear Euler equation of the form

$$((x')^{\alpha})' + \frac{\gamma}{t^{\alpha+1}} x^{\alpha} = 0, \quad x(t_0) = a, \quad x'(t_0) = b, \tag{15}$$

in the case when a is a quotient of two positive odd numbers.

The main goal of this section is to obtain image of Eq. (**15**) under the differential transform at t_0. We apply the formulas introduced in **Section 3**. Number of the applied formula appears in parentheses above the ‘‘=’’ sign.

$$\begin{aligned}
D\{x(t)\} &\overset{(5)}{=} \{X(k)\}_{k=0}^{\infty}, \quad X(0) \overset{(8)}{=} a, \quad X(1) \overset{(8)}{=} b,\\
D\{x'(t)\} &\overset{(9)}{=} \{(k+1)X(k+1)\}_{k=0}^{\infty},\\
D\{(x'(t))^{\alpha}\} &\overset{(12)}{=} \left\{ \sum_{l=1}^{k} \binom{\alpha}{l} (X(1))^{\alpha-l} \hat{B}_{k,l}(2X(2), 3X(3), \dots) \right\}_{k=1}^{\infty}\\
&=: \{H_1(k)\}_{k=1}^{\infty},\\
H_1(0) &= \binom{\alpha}{0} (X(1))^{\alpha},\\
D\{((x'(t))^{\alpha})'\} &\overset{(9)}{=} \{(k+1)H_1(k+1)\}_{k=0}^{\infty},\\
D\{(x(t))^{\alpha}\} &\overset{(12)}{=} \left\{ \sum_{l=1}^{k} \binom{\alpha}{l} (X(0))^{\alpha-l} \hat{B}_{k,l}(X(1), X(2), \dots) \right\}_{k=1}^{\infty}\\
&=: \{H_2(k)\}_{k=1}^{\infty},\\
H_2(0) &= \binom{\alpha}{0} (X(0))^{\alpha},\\
D\left\{ \frac{1}{t^{\alpha+1}} \right\} &\overset{(10)}{=} \left\{ \binom{-\alpha-1}{k} t_0^{-\alpha-1-k} \right\}_{k=0}^{\infty}\\
&=: \{F_1(k)\}_{k=0}^{\infty}.
\end{aligned}$$

Now, we use of the property (**11**) for transforming the product. The image of Eq. (**15**) transformed at $t = t_0$ reads as

$$(k+1)H_1(k+1) + \gamma \sum_{l=0}^{k} F_1(l)H_2(k-l) = 0, \quad k \in \mathbb{N}_0. \tag{16}$$

We demonstrate the use of this result on a concrete example.

Example 4.1. As a testing example, we take the Euler Eq. (**2**) with $\alpha = 3$:

$$((x')^3)' + \frac{\gamma}{t^4}x^3 = 0.$$

We know that if $\gamma = \gamma_3 = \left(\frac{3}{4}\right)^{\cdot}$, then $x_0(t) = t^{3/4}$ is a solution to the initial value problém

$$((x')^3)' + \frac{\left(\frac{3}{4}\right)^4}{t^4}x^3 = 0, \quad x(1) = 1, \quad x'(1) = \frac{3}{4}. \tag{17}$$

For brevity, we will use X(k) instead of $X(k)[1], k \in \mathbb{N}_0$. Transformed Eq. (**16**) expands for (**17**) with

$\alpha = 3, t_0 = 1, \gamma = \left(\frac{3}{4}\right)^4$

$$\begin{aligned}0 &= (k+1)\sum_{l=1}^{k+1}\binom{3}{l}(X(1))^{3-l}\hat{B}_{k+1,l}(2X(2),\ldots)\\ &+ \gamma\sum_{l=0}^{k-1}\binom{-4}{l}\sum_{j=1}^{k-l}\binom{3}{j}(X(0))^{3-j}\hat{B}_{k-l,j}(X(1),\ldots) + \gamma\binom{-4}{k}\binom{3}{0}(X(0))^3,\end{aligned}$$

where the middle term applies only for $k \geq 1$. Taking into account (**13**), we can write the preceding equality in a more compact form

$$\begin{aligned}0 &= (k+1)\sum_{l=1}^{k+1}\binom{3}{l}(X(1))^{3-l}\hat{B}_{k+1,l}(2X(2),\ldots)\\ &+ \gamma\sum_{l=0}^{k}\binom{-4}{l}\sum_{j=0}^{k-l}\binom{3}{j}(X(0))^{3-j}\hat{B}_{k-l,j}(X(1),\ldots), \quad k \in \mathbb{N}_0.\end{aligned}$$

We start with $k = 0$:

$$1 \cdot \binom{3}{1}(X(1))^2\hat{B}_{1,1}(2X(2),\dots) + \gamma\binom{-4}{0}\binom{3}{0}(X(0))^3 = 0$$

$$\Rightarrow 3(X(1))^2 2X(2) + \gamma = 0 \Rightarrow X(2) = \frac{-\gamma}{6(X(1))^2} = \frac{\left(\frac{3}{4}\right)^4}{6\left(\frac{3}{4}\right)^2} = -\frac{3}{32}.$$

Then we substitute $k = 1$:

$$2\left(\binom{3}{1}(X(1))^2\hat{B}_{2,1}(2X(2),\dots) + \binom{3}{2}X(1)\hat{B}_{2,2}(2X(2),\dots)\right)$$
$$+ \gamma\left(\binom{-4}{0}\binom{3}{1}(X(0))^2\hat{B}_{1,1}(X(1),\dots) + \binom{-4}{1}\binom{3}{0}(X(0))^3\right) = 0$$
$$\Rightarrow 2(3(X(1))^2 \cdot 3X(3) + 3X(1)(2X(2))^2) + \gamma(3(X(0))^2X(1) - 4(X(0))^3) = 0$$
$$\Rightarrow X(3) = \cdots = \frac{5}{128}.$$

We continue with $k = 2$:

$$3\Bigg(\binom{3}{1}(X(1))^2\hat{B}_{3,1}(2X(2),\dots) + \binom{3}{2}X(1)\hat{B}_{3,2}(2X(2),\dots)$$
$$+ \binom{3}{3}(X(1))^0\hat{B}_{3,3}(2X(2),\dots)\Bigg)$$
$$+ \gamma\Bigg(\binom{-4}{0}\left[\binom{3}{1}(X(0))^2\hat{B}_{2,1}(X(1),\dots) + \binom{3}{2}X(0)\hat{B}_{2,2}(X(1),\dots)\right]$$
$$+ \binom{-4}{1}\binom{3}{1}(X(0))^2\hat{B}_{1,1}(X(1),\dots) + \binom{-4}{2}\binom{3}{0}(X(0))^3\Bigg) = 0,$$

that is

$$0 = 3\left(3(X(1))^2 4X(4) + 3X(1)2 \cdot 2X(2)3X(3) + (2X(2))^3\right)$$
$$+ \gamma\Big([3(X(0))^2X(2) + 3X(0)(X(1))^2] - 4 \cdot 3(X(0))^2X(1)$$
$$+ \frac{(-4)(-5)}{2}(X(0))^3\Big) \quad \Rightarrow X(4) = \cdots = -\frac{45}{2048}.$$

Now recall that the exact solution to the initial value problem (**17**) is $x0(t) = t^{3/4}$. Taylor series expansion of $t^{3/4}$ at $t_0 = 1$ is

$$1 + \frac{3}{4}(t-1) - \frac{3}{32}(t-1)^2 + \frac{5}{128}(t-1)^3 - \frac{45}{2048}(t-1)^4 + O((t-1)^5).$$

Remark 4.2.

A different approach to find recurrence relations for obtaining the coefficients of a Taylor series of the solution is presented in papers about the Parker-Sochacki method (see for example [**30,33**]). The main idea is to transform an ordinary differential equation into a system of first order ordinary differential equations with nothing but polynomials on the righthand side. Such a method can be applied to a wide class of ordinary differential equations. However, describing the process of finding the polynomial form is not simple and the polynomial system might not be unique. A-priori error estimates presented in [**30**] can be applied especially to this polynomial form of the equation. Within this context, choosing the transformations $y_1 = x(t), y_2 = x'(t), y_3 = t^{-4}, y_4 = t, y_5 = \frac{1}{x'(t)}$, the initial value problem (**17**) can be rewritten in the following polynomial form:

$$\begin{aligned}
&y_1' = y_2, && y_1(1) = 1,\\
&y_2' = -\frac{1}{3}\left(\frac{3}{4}\right)^4 y_1^3\, y_3\, y_5^2, && y_2(1) = \frac{3}{4},\\
&y_3' = -4y_3^2\, y_4^3, && y_3(1) = 1,\\
&y_4' = 1, && y_4(1) = 1,\\
&y_5' = \frac{1}{3}\left(\frac{3}{4}\right)^4 y_1^3\, y_3\, y_5^4, && y_5(1) = \frac{4}{3}.
\end{aligned}$$

5. Application of the differential transform to the Euler equation with a proportional delay

Consider the initial value problém

$$((x'(t))^{\alpha})' + \frac{\gamma}{t^{\alpha+1}}(x(\lambda t))^{\alpha} = 0, \quad x(t_0) = a, \quad x'(t_0) = b, \tag{18}$$

with λ ∈ (0,1) and the initial function

$$\phi(t) = a + b(t - t_0), \quad t \in (0, t_0] = I_0. \tag{19}$$

Let $t_i = \frac{t_0}{\lambda^i}$ and $I_i = [t_{i-1}, t_i]$, $i \in \mathbb{N}_0$. If $t \in I_i$

If $t \in I_i$ then λt lies in $I_i - 1$. We follow the process of combining the differential transform with the method of steps for delayed differential equations described in the paper [**28**].

For $t \in I_1 = [t_0, t_1]$ we determine the solution of the initial value problem (**18**), (**19**) as $x_1(t)$. The differential transform of $x_1(t)$ at t_0 will be $X_1(k)[t_0]$, $k \in \mathbb{N}_0$. Since λt for $t \in I_1$ falls into I_0, we substitute the initial function (**19**) for $x(\lambda t)$ and rewrite Eq. (**18**) in the form

$$((x'(t))^{\alpha})' + \frac{\gamma}{t^{\alpha+1}}(a + b(\lambda t - t_0))^{\alpha} = 0, \quad x(t_0) = a, \quad x'(t_0) = b. \tag{20}$$

Because

$$\phi(\lambda t) = a + b(\lambda t - t_0) = a + bt_0(\lambda - 1) + b\lambda(t - t_0),$$

and the fcth component of the differential transform of $(\phi(\lambda t))^{\alpha}$ at t_0 is

$$D\{(\phi(\lambda t))^{\alpha}\}(k)[t_0] \overset{(12)+(13)}{=} \sum_{l=0} \binom{\alpha}{l}(a+bt_0(\lambda-1))^{\alpha-l}\hat{B}_{k,l}(b\lambda,0,0,\ldots)$$
$$= \binom{\alpha}{k}(a+bt_0(\lambda-1))^{\alpha-k}(b\lambda)^k,\ k\geq 0.$$

After we use the formula (**11**), Eq. (**20**) transformed at t_0 reads as

$$0=(k+1)\sum_{l=1}^{k+1}\binom{\alpha}{l}(X_1(1)[t_0])^{\alpha-l}\hat{B}_{k+1,l}(2X_1(2)[t_0],3X_1(3)[t_0]\ldots) \qquad (21)$$
$$+\gamma\sum_{l=0}^{k}\binom{-\alpha-1}{l}(t_0)^{-\alpha-1-l}\binom{\alpha}{k-l}(a+bt_0(\lambda-1))^{\alpha-k+l}(b\lambda)^{k-l}.$$

The initial conditions are transformed to

$$X_1(0)[t_0]=a, \quad X_1(1)[t_0]=b.$$

Substitution for $k = 0,1, \ldots$ into (**21**) provides recurrence relations from which one can successively calculate $X_1(k)[t_0]$ for $k \geq 2$. The solution on the interval I_1 is then

$$x_1(t)=\sum_{k=0}^{\infty}X_1(k)[t_0](t-t_0)^k.$$

In applications, we use a computing software to calculate the coefficients of the Taylor series. It means that this series as well as any other series will be truncated. The solution x_1 will become an approximate solution.

Notice that with a general initial function ϕ,, Eq. (**21**) would have the form

$$(k+1)\sum_{l=1}^{k+1}\binom{\alpha}{l}(X_1(1)[t_0])^{\alpha-l}\hat{B}_{k+1,l}(2X_1(2)[t_0],3X_1(3)[t_0]\ldots)$$
$$+\gamma\sum_{l=0}^{k}\binom{-\alpha-1}{l}(t_0)^{-\alpha-1-l}D\{(\phi(\lambda t))^{\alpha}\}(k)[t_0]=0.$$

Now we proceed with the second step. Take $t \in I_2 = [t_1, t_2]$, denote $x_2(t)$the approximate solution on I_2, and let $X_2(k)[t_1]$ for $k \geq 0$ be the differential transform of x_2 at t_1. Since λt for $t \in I_2$ lies in I_1, we substitute for $x(\lambda t)$ the function $x_1(\lambda t)$ and rewrite Eq. (**18**) in the form

$$((x'(t))^{\alpha})'+\frac{\gamma}{t^{\alpha+1}}(x_1(\lambda t))^{\alpha}=0, \quad x(t_1)=x_1(t_1), \quad x'(t_1)=x_1'(t_1). \qquad (22)$$

Since $\lambda t = \lambda t_1 + \lambda(t - t_1)$, we have

$$\mathcal{D}\{\lambda t\} = \{\lambda t_1, \lambda, 0, 0, \dots\}.$$

Recalling that $\lambda t_1 = t_0$, we have the following expression for the kth component of the differential transform of $x_1(\lambda t)$ at t_1:

$$\begin{aligned}\mathcal{D}\{x_1(\lambda t)\}(k)[t_1] &\overset{(12)}{=} \sum_{l=1}^{k} X_1(l)[\lambda t_1]\hat{B}_{k,l}(\lambda, 0, 0, \dots) = X_1(k)[t_0]\lambda^k \\ &=: G(k)[t_1], k \geq 1, \\ \mathcal{D}\{x_1(\lambda t)\}(0)[t_1] &= X_1(0)[t_0] = a =: G(0)[t_1].\end{aligned}$$

Next, for $(x_1(\lambda t))^\alpha$ we get

$$\begin{aligned}&\mathcal{D}\{(x_1(\lambda t))^\alpha\}(k)[t_1] \\ &\overset{(12)+(13)}{=} \sum_{l=0}^{k} \binom{\alpha}{l} (G(0)[t_1])^{\alpha-l} \hat{B}_{k,l}(G(1)[t_1], G(2)[t_1], \dots), k \geq 0.\end{aligned}$$

Again, we use the product formula (**11**) and Eq. (**22**) transformed at reads as

$$\begin{aligned}0 = (k+1) &\sum_{l=1}^{k+1} \binom{\alpha}{l} (X_2(1)[t_1])^{\alpha-l} \hat{B}_{k+1,l}(2X_2(2)[t_1], 3X_2(3)[t_1] \dots) \\ &+ \gamma \sum_{l=0}^{k} \binom{-\alpha-1}{l} (t_1)^{-\alpha-1-l} \\ &\times \sum_{j=0}^{k-l} \binom{\alpha}{j} (G(0)[t_1])^{\alpha-j} \hat{B}_{k-l,j}(G(1)[t_1], G(2)[t_1], \dots).\end{aligned} \tag{23}$$

According to (**14**), the initial conditions are

$$\begin{aligned}X_2(0)[t_1] &= X_1(0)[t_1] = \sum_{k=0}^{\infty} X_1(k)[t_0](t_1 - t_0)^k, \\ X_2(1)[t_1] &= X_1(1)[t_1] = \sum_{k=0}^{\infty} (k+1) X_1(k+1)[t_0](t_1 - t_0)^k.\end{aligned}$$

The approximate solution x_2 for $t \in I_2$ is then

$$x_2(t) = \sum_{k=0}^{\infty} X_2(k)[t_1](t - t_1)^k.$$

Further steps for $t \in I_i$, $i \geq 3$ lead again to the recurrence relation (**23**). The only differences will appear in indices of the Taylor coefficients (that is, X_2 becomes X_i) and centres of the Taylor expansion (that is, t_1 becomes t_{i-1}). The process of calculation of the Taylor coefficients in the ith step follows the pattern of the second step.

Example 5.1. To demonstrate the described algorithm, we choose the following concrete values of the parameters:

$$\alpha = 3, \quad t_0 = 1, \quad \gamma = \left(\frac{3}{4}\right)^4, \quad a = 1, \quad b = \frac{3}{4}, \quad \lambda = 0.8. \tag{24}$$

Notice that the constants α, t_0 and γ have the exactly same values as in **Example 4.1**. Comparison with the solution $x_0(t) = t^{3/4}$ to the non-delayed problem (**17**) studied in **Example 4.1** will allow us to observe the effect of the delay and the chosen initial function. This is important because it is not possible to find exact solution to the initial value problem (**18**), (**19**) with the constants (**24**) in terms of elementary functions.

The Eq. (**18**) with the constants (**24**) becomes

$$((x'(t))^3)' + \left(\frac{3}{4}\right)^4 \frac{1}{t^4}(x(0.8t))^3 = 0, \quad x(1) = 1, \quad x'(1) = \frac{3}{4}, \tag{25}$$

and the initial function is

$$\phi(t) = 1 + \frac{3}{4}(t-1), \quad t \in (0, 1]. \tag{26}$$

Recalling the fact that $t_i = \frac{t_0}{\lambda i'}$, the first step of the algorithm takes place on the interval $[t_0, t_1] = [1, \frac{10}{8}]$ whereas the second step takes place on the interval $[t_1, t_2] = [\frac{10}{8}, \frac{100}{64}]$. The results of the simulation in Matlab, version 2019b, are shown in **Table 1** and **Fig. 1**. The first column of **Table 1** presents the values of t in the interval $[t_0, t_2]$ where the comparison is done. In the second column we have values of the approximate solution to the IVP (**25**), (**26**) found by using the differential transform algorithm. Here x_1 represents the solution on $[t_0, t_1]$ and x_2 the solution on $[t_1, t_2]$. In both cases, the order of the Taylor polynomial that represents the approximate solution is 5. We note that the accuracy influenced by the chosen order is good enough. The difference between the 5th order approximate solution and higher order approximate solutions on the observed interval is less than 10^{-6}. The third column contains values of the approximate solution computed by the built-in Matlab function designed for solving delay differential equations ''ddesd''. The fourth column shows the difference between these two numerical solutions at given points. In the fifth column we present values of the exact solution to the initial value problem (**17**), that is, to the non-delayed half-linear Euler equation. At the end of the interval $[t_0, t_2]$, we can observe that the solution to the half-linear Euler equation with delay tends to grow faster than the solution to the half-linear Euler equation without delay. This fact is illustrated also in **Fig. 1**.

Table 1 Comparison of DT (order 5) and Matlab.

	$x_1(x_2)$	x_{ddesd}	$x_{ddesd} - x_i$	$x_0 = t^{3/4}$
$t_0 = 1.0000$	1.0000	1.0000	0.0	1.0000
1.0370	1.0621	1.0621	0.0147E–3	1.0277
1.0741	1.1234	1.1235	0.1103E–3	1.0551
$t_1 \approx 1.1111$	1.1840	1.1843	0.3515E–3	1.0822
1.1523	1.2591	1.2595	0.0004	1.1122
1.1934	1.3340	1.3339	−0.0002	1.1418
$t_2 \approx 1.2346$	1.4089	1.4075	−0.0013	1.1712

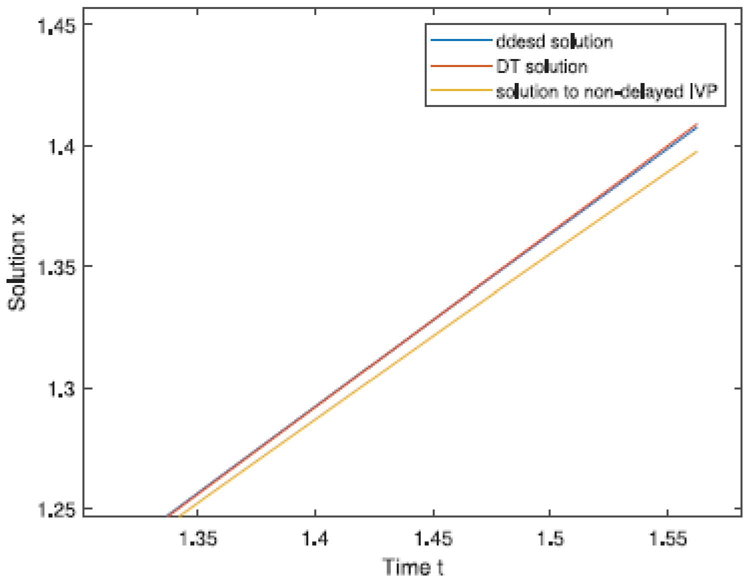

Fig. 1. Comparison of approximate solutions to (**25**), (**26**) and the exact solution to (**17**).

Experimenting with other values of y, we have noticed that neither ''ddesd' nor our algorithm can successfully calculate the approximate solution on the interval $[t_0,t_2]$ if y increases to 1.9. Similar situation happens if we keep y at the constant value 1 and try to calculate on the interval $[t_0, 3.0]$: Matlab returns an error message. Such behaviour suggests that more research in this direction is needed.

6. Conclusion

Half-linear differential equations are the subject of ongoing research, especially in the field of asymptotic behaviour and qualitative properties. However, the search for suitable numerical algorithms, has not yet been considered. e presented how the differential transform algorithm can be applied to obtain numerical solutions to second-order half-linear Euler equations without delay and with a proportional delay. The described algorithm includes a modification for different types of initial functions. Applicability of the algorithm is demonstrated first on an example of the Euler equation without delay. The coefficients of the Taylor expansion obtained by the differential transform coincide with coefficients of Taylor expansion of the exact solution. Numerical simulation on an example of the delayed equation with concrete values of parameters was performed. The simulation values are in a good agreement with simulations produced by the Matlab routine ''ddesd''. All obtained results confirm that the presented algorithm is efficient and convenient for finding approximate solutions to the studied initial value problems.

Our experiment shows that the differential transform method in combination with the method of steps is well applicable to delayed half-linear Euler equations. One of the advantages is that the obtained approximate solution is in the form of a Taylor polynomial. That is different from the outcome of the Matlab function ''ddesd'', where the result is a set of approximate function values of the solution. We also found out that both ''ddesd'' function and our procedure do not work on larger intervals in some cases, particularly with increasing y. On the other hand, the combination of the differential transform and the method of steps can be theoretically elongated easily. The practical implementation of the algorithm, however, has to deal with the limits of division by numbers close to zero, especially when increasing the order of the Taylor polynomial. The idea of scaling the coefficients to the same dimension by the length of the considered interval might help to overcome this obstacle.

Finally, the knowledge of approximate solutions to initial value problems for half-linear Euler type equations can complement the qualitative theory and motivate further theoretical results. Lat but not least, the lack of success with computations on larger intervals using Matlab gives a strong motivation for continuing research on approximate solutions.

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