Value Set-Based Numerical Analysis of Robust Stability for Fractional-Order Retarded Quasi-Polynomials with Uncertain Parameters and Uncertain Fractional Orders

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Abstract. This example-oriented contribution deals with the value set-based numerical analysis of robust stability for the family of fractional-order retarded quasi-polynomials with both uncertain parameters and uncertain fractional orders. The specific investigated feedback control system consists of the fractional-order PID controller and the controlled plant, represented by a heat transfer process described by the linear time-invariant fractional-order time-delay model with parametric uncertainty (with three uncertain parameters, namely, gain, fractional time constant, and fractional time-delay term, and furthermore two fractional orders). The graphical robust stability analysis is based on the numerical calculation of the value sets and the application of the zero exclusion principle.

Keywords: Robust Stability, Fractional-Order Systems, Parametric Uncertainty, Retarded Quasi-Polynomials, Uncertain Fractional Orders, Value Set, Zero Exclusion Principle.

1 Introduction

Stability is the most important property of all control systems, and its preservation represents a crucial task in the control system design practice. However, stability needs to be ensured not only for the idealized case with a nominal mathematical model of a controlled process but also for the more realistic scenario, i.e., under conditions of uncertainty. Thus, we are frequently interested in robust stability rather than just in (nominal) stability. From the variety of possible uncertain models, the systems with parametric uncertainty belong among the most popular and comprehensible ones [1], [2], especially under the assumption of a box-shaped uncertainty bounding set.

The application of fractional calculus in the field of control engineering has gained the increased attention of many researchers during approximately the last two decades

[3-5] because of the improved performance of fractional-order control systems in comparison with the conventional integer-order ones. Not surprisingly, the robust stability of fractional-order systems under various uncertainty conditions is in line with this positive trend [6, 7]. The investigation of robust stability for systems with parametric uncertainty becomes a complicated task for increasing the complexity of uncertainty structure in the analyzed family of (quasi-)polynomials, even for the integer-order case. Moreover, the fractional orders further increase the intricacy of the robust stability analysis problem [6, 8]. It was already shown in the literature that the value set concept in combination with the zero exclusion principle represents a powerful numerical tool for the investigation of robust stability, especially for linear time-invariant (LTI) fractional-order systems with complicated uncertainty structures [8]. Furthermore, the same graphical test is also applicable to families of retarded quasi-polynomials [8, 9]. Besides, an interesting problem occurs if the systems have crisply known fractional orders [10].

This example-oriented contribution deals with the analysis of robust stability for the family of fractional-order retarded quasi-polynomials with both uncertain parameters and uncertain fractional orders. The feedback control system consists of the fractional-order PID controller and the controlled plant, represented by a heat transfer process described by the LTI fractional-order time-delay model with parametric uncertainty (with three uncertain parameters, to wit, gain, fractional time constant [11], and fractional time-delay term [12], and additionally two fractional orders [10]). This combination leads to the family of closed-loop characteristic fractional-order retarded quasi-polynomials with uncertain parameters and uncertain fractional orders. The robust stability test itself is based on the numerical calculation of the value sets and the application of the zero exclusion principle [2, 8]. The value sets are obtained via sampling (gridding) the uncertain parameters and fractional orders and subsequent direct calculation of related partial points of the value sets for a supposed frequency range.

2 Mathematical Description of A Heat Transfer Process

Suppose a heat transfer process for the heat flux loss case, which was mathematically described in [12, 13] by means of the fractional-order partial differential equation. The same works [12, 13] also derived the transfer function describing the temperature at the point λ_1 with respect to the heat flux at the beginning of the beam. This transfer function has the form [12, 13]:

$$G(\lambda_1, s) = \frac{T(\lambda_1, s)}{H(0, s)} = \frac{K}{T_1 s^{0.5} + 1} e^{-\Theta s^{0.5}} \tag{1}$$

where the experimentally obtained parameters from [12] are $K = 3.2$, $T_1 = 209.27$, and $\Theta = 2.97$.

In [14], it was assumed that three parameters K, T_1, and Θ might vary within ±10% of their nominal values from [12]. In this contribution, we will suppose that not only these parameters but also fractional orders can vary within given bounds that correspond to ±10% of the nominal values. That means that the heat transfer process is described by the parametrically uncertain fractional-order time-delay model:

$$G(\lambda_1, s, K, T_1, \Theta, \alpha, \beta) = \frac{T(\lambda_1, s)}{H(0, s)} = \frac{K}{T_1 s^{\alpha} + 1} e^{-\Theta s^{\beta}} \tag{2}$$

where:

$$\begin{aligned} K &\in [2.88,\, 3.52] \\ T_1 &\in [188.343,\, 230.197] \\ \Theta &\in [2.673,\, 3.267] \\ \alpha &\in [0.45,\, 0.55] \\ \beta &\in [0.45,\, 0.55] \end{aligned} \tag{3}$$

The structure of the model (2) and the uncertainty bounding set (3) form the family of controlled plants that will be used in the following analysis.

3 Fractional-Order PID Controller

In [12], several integer-order and fractional-order PID controllers were designed for the nominal system (1) and compared mutually. The fractional-order PID controller:

$$C(s) = 259.45\left(1 + \frac{1}{67.74 s^{0.5}} + 0.18 s^{0.5}\right) \tag{4}$$

with fixed integration and differentiation orders 0.5 and with the parameters obtained by minimization of ITAE criterion seems to represent a very good compromise between the performance indicators and minor steady-state oscillations [12].

4 Value Set-Based Robust Stability Analysis

The family of closed-loop characteristic fractional-order retarded quasi-polynomials, which corresponds to the feedback control system with the plant family (2), (3) and the controller (4), is:

$$\begin{aligned} p_{CL}(s, K, T_1, \Theta, \alpha, \beta) &= 67.74 T_1 s^{\alpha} s^{0.5} + 67.74 s^{0.5} + \cdots \\ &\cdots + 3163.52574 K s^{0.5} s^{0.5} e^{-\Theta s^{\beta}} + 17575.143 K s^{0.5} e^{-\Theta s^{\beta}} + 259.45 K e^{-\Theta s^{\beta}} \end{aligned} \tag{5}$$

where the parameters K, T_1, Θ, and fractional orders α and β may vary in accordance with (3). For the remainder of this section, they all will be called simply "uncertain parameters".

In some specific cases, the value sets of the fractional-order families can be constructed on the basis of analysis of vertices and exposed edges [15-18]. Nevertheless, in this contribution, the value sets for the family (5) with relatively complicated uncertainty structure, including uncertain fractional orders, are plotted by means of sampling (gridding) the uncertain parameters and subsequent direct calculation of related partial points of the value sets for a supposed frequency range.

The value sets of the family (5) for the frequencies from 0 to 10 with step 0.2 are plotted in Fig. 1. All 5 uncertain parameters are sampled by 5 equidistant values, i.e., each value set for a fixed frequency consists of $5^5 = 3125$ points. The full view from Fig. 1 provides an overview of value sets' shape. Then, a closer view of the situation in the neighborhood of the complex plane origin (zero point) can be seen in Fig. 2, where the value sets are plotted for the frequencies from 0 to 2 with step 0.02.

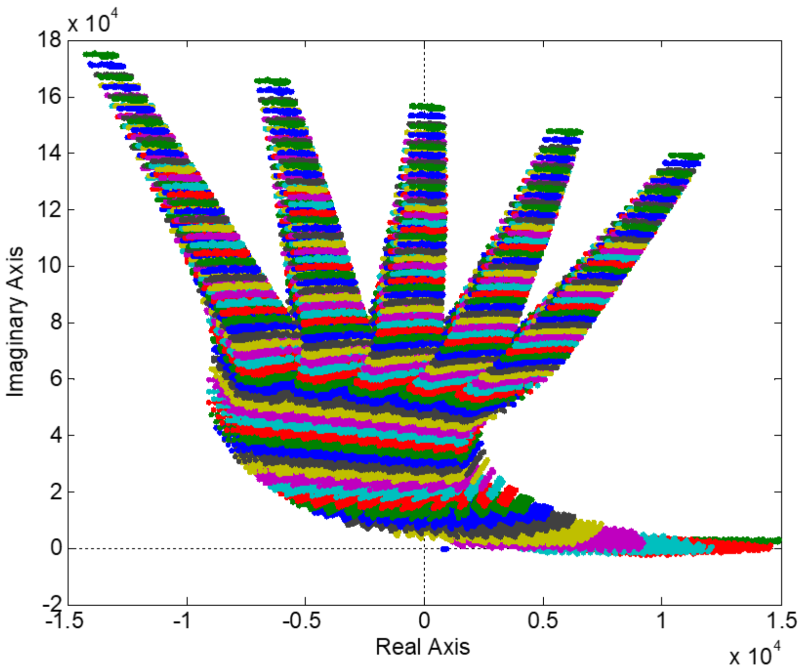

Fig. 1. The value sets of the family (5) for the frequencies from 0 to 10.

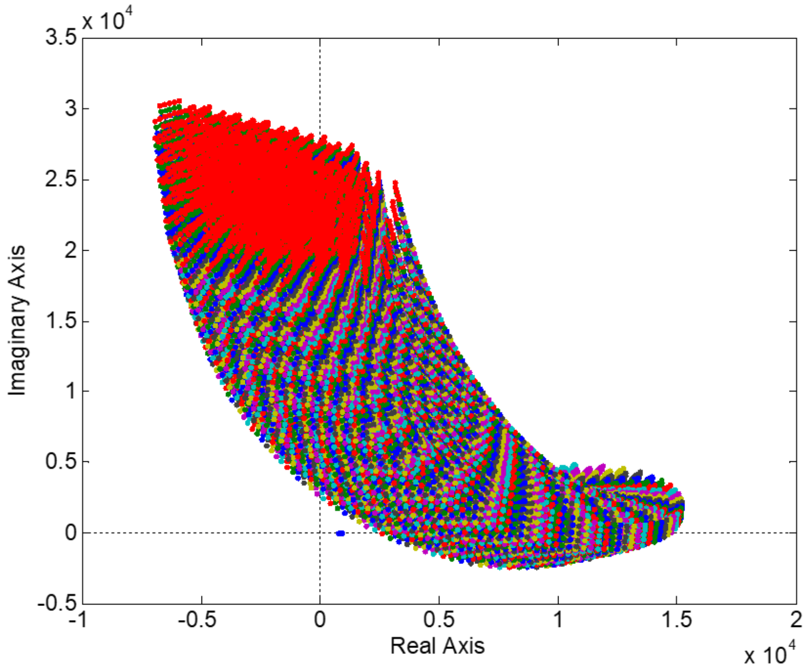

Fig. 2. The value sets of the family (5) for the frequencies from 0 to 2.

In accordance with the zero exclusion principle [2, 8], the family of fractional-order retarded quasi-polynomials (5) is robustly stable if and only if the zero point is excluded from the value sets. However, the correct application of the zero exclusion principle is generally preconditioned by several requirements, which include the invariant degree of the (quasi-)polynomial family, pathwise connected uncertainty bounding set, continuous coefficient functions, and at least one stable member of the family. One can observe that the value sets are excluded, and the majority of preconditions are fulfilled (e.g., the existence of a stable member can be verified by the generalization of modified Mikhailov stability criterion from [9]). Nevertheless, the invariant degree condition seems unfulfilled since the degree of the term $67.74T_1s^{\alpha}s^{0.5}$ in (5) may change depending on the actual value of the fractional order $\alpha \in [0.45, 0.55]$. (Note that the validity of the index law is not so straightforward in fractional calculus as in the integer-order case [19, 20]). Anyway, the family (5) is robustly stable according to the zero exclusion principle despite the possible change of the retarded quasi-polynomial degree.

Altogether, the family (5), and thus also the feedback control loop with the plant (2) and controller (4), is robustly stable, i.e., it remains stable for all possible variations of the uncertain parameters (3).

5 Conclusion

This contribution was focused on the value set-based numerical analysis of robust stability for the family of fractional-order retarded quasi-polynomials with both uncertain parameters and uncertain fractional orders. The elaborated experiment indicates that the value set-based approach is applicable even to these complex families, but further research will be needed, especially more thorough research on the impact of the invariant degree condition on the application of the zero exclusion principle to fractional-order systems.

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