NONOSCILLATORY SOLUTIONS OF HALF-LINEAR EULER-TYPE EQUATION WITH n TERMS

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ABSTRACT. We consider the half-linear Euler-type equation with n terms

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \sum_{j=1}^{n-1} \frac{\mu_p}{t^p \operatorname{Log}_j^2 t} + \frac{\mu}{t^p \operatorname{Log}_n^2 t}\right)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-1} \operatorname{sgn} x$$

in the subcritical case when $0 < \mu < \mu_p$ and $p > 1$. The solutions of this nonoscillatory equation cannot be found in an explicit form and can be studied only asymptotically. In this paper, with the use of the perturbation principle, modified Riccati technique and the fixed point theorem, we establish an asymptotic formula for one of its solutions.

1. INTRODUCTION

The aim of the paper is to find an asymptotic formula for a nonoscillatory solution of half-linear Euler-type differential equation

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \sum_{j=1}^{n-1} \frac{\mu_p}{t^p \operatorname{Log}_j^2 t} + \frac{\mu}{t^p \operatorname{Log}_n^2 t}\right)\Phi(x) = 0 \tag{1}$$

for $0 < \mu < \mu_p$, $t \in [T, \infty)$. The operator Φ is defined as $\Phi(x) := |x|^{p-1} \operatorname{sgn} x$, $p > 1$, $n \in \mathbb{N}$, γ_p and μ_p are the constants

$$\gamma_p := \left(\frac{p-1}{p}\right)^p, \quad \mu_p := \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}$$

and $\operatorname{Log}_j t$ are products of iterated logarithmic functions:

$$\operatorname{Log}_j t := \prod_{k=1}^{j} \log_k t, \ \ \log_1 t := \log t, \ \log_k t := \log_{k-1}(\log t), \ k \geq 2.$$

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The studied equation (1) is a special case of a general half-linear second order differential equation

$$L[x] := (r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \tag{2}$$

where $x = x(t)$, functions $r(t), c(t)$ are continuous and $r(t)$ is positive on the interval of consideration. The solution space of (2) is linear (but not additive) and contains either oscillatory or nonoscillatory solutions. Within the studies of equation (2) in some neighborhood of infinity, i.e., $t \in [T, \infty)$ for some T, its solution can be classified as oscillatory, if it has got infinitely many zeros tending to infinity, and nonoscillatory otherwise. Oscillatory and nonoscillatory solutions cannot coexist, hence half-linear equations are said to be oscillatory or nonoscillatory according to behavior of their all solutions (for more information see the basic literature [3] summing up the results for half-linear equations up to the year 2005).

Describing asymptotic properties of solutions is one of the main tasks of qualitative theory of differential equations. Asymptotic behavior of solutions is often being classified with the use of the theory of regularly varying functions in the sense of Karamata. Let us recall the following notation (for more information see for example the monograph [10]).

A measurable function $f : [a, \infty) \to (0, \infty)$ is called regularly varying (at infinity) of index ϑ (and we write $f \in RV(\vartheta)$) if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\vartheta} \quad \text{for every } \lambda > 0.$$

If $\vartheta = 0$, f is called slowly varying. The Representation Theorem (see for example [14]) says that $f \in RV(\vartheta)$ if and only if it can be expressed in the form

$$f(t) = \varphi(t) t^{\vartheta} \exp\left\{ \int_a^t \frac{\psi(s)}{s} \, \mathrm{d}s \right\},$$

where $t \geq a$ for some $a > 0$, φ and ψ are measurabe functions such that $\lim_{t\to\infty} \varphi(t)$ is finite and positive and $\lim_{t\to\infty} \psi(t) = 0$.

Half-linear equations have been studied in the framework of regularly varying functions for example by the group of authors Jaroš, Kusano, Manojlović, Marić, Tanigawa and Řehák, see [7, 8, 9, 14] and references therein. Considering the latest papers which sum up, improve and extend previous results in this field, Řehák in [14] provided an exhaustive overview of asymptotic formulas for (normalized) regularly varying solutions of (2) in the case when r is positive and c negative on $[T, \infty)$. Furthermore, Manojlović and Kusano in [8] established asymptotic formulas for nonoscillatory solutions of (2) depending on the rate of decay toward zero of the positive function

$$Q_C = t^{p-1} \int_t^{\infty} c(s) \, \mathrm{d}s - C$$

as $t \to \infty$, where $C < \frac{\gamma_p}{p-1}$. However, the results are inapplicable to our equation (1), because the constant C in this case does not satisfy strict inequality, but equality. The reason is that equation (1) lies on the threshold between oscillation and nonoscillation. For such equations it was shown for example in [11] that it can be useful to regard the studied equation as a perturbation of a nonoscillatory equation with less terms.

Equation (1) can be seen as a perturbation and in some sense also a generalization of the half-linear Euler equation

$$(\Phi(x'))' + \frac{\gamma}{t^p}\Phi(x) = 0. \tag{3}$$

The critical coefficient $\gamma = \gamma_p$ is its oscillation constant: replaced by a greater constant, the Euler equation becomes oscillatory, and for γ_p and smaller constants it is nonoscillatory. In view of this property, one can say that the Euler equation is conditionally oscillatory. Equation (3) in the case $\gamma = \gamma_p$ has a pair of linearly independent nonoscillatory solutions (see for example [3], Chapter 1.4.2)

$$x_1(t) = t^{\frac{p-1}{p}}, \quad x_2(t) = t^{\frac{p-1}{p}} \log^{\frac{2}{p}}(t)(1+o(1)) \quad \text{as } t \to \infty.$$

If $\gamma < \gamma_p$ then (3) has a pair of solutions

$$x_{1,2} = t^{\lambda_{1,2}^{q-1}},$$

where $\lambda_{1,2}$ are zeros of $|\lambda|^q - \lambda + \frac{\gamma}{p-1} = 0$, as can be seen by a direct substitution.

The Euler equation (3) with the critical constant can be perturbed so that the resulting equation is again conditionally oscillatory. Such a one term perturbation leads to the half-linear Riemann–Weber equation

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \frac{\mu}{t^p \log^2 t}\right)\Phi(x) = 0, \tag{4}$$

whose oscillation constant is $\mu = \mu_p$. In this critical case for $\mu = \mu_p$ equation (4) is nonoscillatory and possesses a pair of linearly independent solutions (see [5])

$$x_1(t) = t^{\frac{p-1}{p}} \log^{\frac{1}{p}} t(1+o(1)), \quad x_2(t) = t^{\frac{p-1}{p}} \log^{\frac{1}{p}} t \log^{\frac{2}{p}}(\log t)(1+o(1)) \quad \text{as } t \to \infty.$$

In the subcritical case for $0 < \mu < \mu_p$ the asymptotic formulas of a pair of linearly independent solutions were found in [11], namely

$$x_{1,2}(t) = t^{\frac{p-1}{p}} (\log t)^{\nu_{1,2}}(1+o(1)) \quad \text{as } t \to \infty,$$

where $\nu_{1,2} = \frac{1}{p}\left(\frac{p-1}{p}\right)^{1-p} \lambda_{1,2}$ and $\lambda_{1,2}$ are zeros of $\frac{\lambda^2}{4\mu_p} - \lambda + \mu = 0$. These formulas can be obtained also as a special case of more general results proved in [1] (a link between these two approaches can be found in [13]).

Finally, the Euler-type equation (also called the generalized Riemann-Weber equation) with an arbitrary number of perturbation terms (1) was studied in [5] and it was shown that its oscillation constant is again $\mu = \mu_p$ and that the equation

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \sum_{j=1}^{n} \frac{\mu_p}{t^p \operatorname{Log}_j^2 t}\right)\Phi(x) = 0 \tag{5}$$

has a pair of linearly independent solutions

$$x_1(t) = t^{\frac{p-1}{p}} \operatorname{Log}_n^{\frac{1}{p}} t(1+o(1)), \quad x_2(t) = t^{\frac{p-1}{p}} \operatorname{Log}_n^{\frac{1}{p}} t \log_{n+1}^{\frac{2}{p}} t(1+o(1)) \quad \text{as } t \to \infty.$$

Notice that all the above solutions are regularly varying of certain indexes and the functions of the form $(1+o(1))$ are slowly varying functions (see [11, 12]).

In this paper we consider the subcritical case of (1) when $0 < \mu < \mu_p$ and reveal asymptotics of one of its solutions. Our motivation comes among others from [13], where the asymptotic formulas for solutions of (1) were proposed in the case $n = 2$. Let us point out that also further generalizations of (1) are a subject of recent studies, see [1, 6] and references given therein.

2. Preliminaries

An important role in the proof of the main result is played by the so-called Riccati technique, which is based on the following facts (see for example [3]). If $x(t) \neq 0$ is a solution of (2) on I, then $w = r(t)\Phi\left(\frac{x'(t)}{x(t)}\right)$ is a solution of the half-linear Riccati equation

$$w(t) + c(t) + (p-1)r^{1-q}(t)|w(t)|^q = 0, \quad q = \frac{p}{p-1}. \tag{6}$$

Within the perturbation approach, the Riccati equation (6) is insufficient for our purposes. The way how to employ the idea of perturbation in the Riccati equation is in making its following modification. Denote for a positive differentiable function $h(t)$

$$R(t) = r(t)h^2(t)|h'(t)|^{p-2}, \quad G(t) = r(t)h(t)\Phi(h'(t)). \tag{7}$$

Then $v(t) = h^p(t)w(t) - G(t)$ is a solution of the so-called modified Riccati equation (see for example [2])

$$v'(t) + h(t)L[h](t) + (p-1)r^{1-q}(t)h^{-q}(t)H(v,G) = 0, \tag{8}$$

where

$$H(v,G) = |v+G|^q - q\Phi^{-1}(G)v - |G|^q.$$

According to [1], equation (8) is in some sense close to the so-called approximate Riccati equation

$$u'(t) + h(t)L[h](t) + \frac{q}{2R(t)}u^2(t) = 0, \tag{9}$$

where the nonlinearity is only quadratic. The estimates of how close to each other are the solutions of (8) and (9) can be done with the use of the calculations from the proof of Theorem 2 in [4] (see also [1], Lemma 1), which we formulate here in the case of our interest for $r(t) = 1$, $h(t) = t^{\frac{p-1}{p}}$, $G = 2\mu_p = \Gamma_p$, $R = \left(\frac{p-1}{p}\right)^{p-2} t$.

Lemma 2.1. *If* $\varepsilon \in (0,1)$ *and*

$$b(\varepsilon) = \begin{cases} \left|\frac{q(q-2)}{6}\right|(1+\varepsilon)^{q-3} & \textit{for} \quad q \geq 3, \\ \left|\frac{q(q-2)}{6}\right|(1-\varepsilon)^{q-3} & \textit{for} \quad q < 3, \end{cases}$$

then for $|v/2\mu_p| < \varepsilon$

$$\left|\frac{(p-1)}{t}H(v,\Gamma_p) - \frac{1}{4\mu_p t}v^2\right| \leq \frac{Lb(\varepsilon)}{t}|v|^3, \tag{10}$$

where $L = \left(\frac{p-1}{p}\right)^{3-2p}$.

3. Main result

The main result of our paper providing an asymptotic formula of a solution of (1) reads as follows.

Theorem 3.1. *Equation* (1) *with* $\mu \in (0, \mu_p)$ *has a solution of the form*

$$x(t) = t^{\frac{p-1}{p}} \operatorname{Log}_{n-1}^{\frac{1}{p}} t \log_n^{\frac{2\lambda}{p}} t(1 + o(1)) \quad \text{as} \quad t \to \infty, \tag{11}$$

where $\lambda = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{\mu}{\mu_p}}$.

Proof. We consider equation (1) to be a perturbation of the Euler equation (3), which is the reason for choosing its solution as the function h in the modified Riccati equation (8). The modified Riccati equation (8) for (1) with

$$h(t) = t^{\frac{p-1}{p}}, G(t) = \Gamma_p = 2\mu_p \tag{12}$$

reads as

$$v'(t) + \sum_{j=1}^{n-1} \frac{\mu_p}{t \operatorname{Log}_j^2 t} + \frac{\mu}{t \operatorname{Log}_n^2 t} + \frac{p-1}{t} H(v(t), \Gamma_p) = 0 \tag{13}$$

and the approximate Riccati equation (9) is the in the form

$$u'(t) + \sum_{j=1}^{n-1} \frac{\mu_p}{t \operatorname{Log}_j^2 t} + \frac{\mu}{t \operatorname{Log}_n^2 t} + \frac{1}{4\mu_p t} u^2(t) = 0. \tag{14}$$

One can see that, with the use of the substitution $u = 4\mu_p t \frac{y'}{y}$, this is the classical Riccati equation of the linear second order equation

$$(4\mu_p t y'(t))' + \left(\sum_{j=1}^{n-1} \frac{\mu_p}{t \operatorname{Log}_j^2 t} + \frac{\mu}{t \operatorname{Log}_n^2 t} \right) y(t) = 0,$$

which can be rewritten, using the notation $\tau = \frac{\mu}{\mu_p}$, as

$$(t y'(t))' + \left(\sum_{j=1}^{n-1} \frac{1}{4t \operatorname{Log}_j^2 t} + \frac{\tau}{4t \operatorname{Log}_n^2 t} \right) y(t) = 0. \tag{15}$$

The transformation of variable $\frac{1}{t}\,\mathrm{d}t = \mathrm{d}s, t = e^s, s = \log t$ leads to the equation

$$y''(s) + \left(\frac{1}{4s^2} + \sum_{j=1}^{n-2} \frac{1}{4s^2 \operatorname{Log}_j^2 s} + \frac{\tau}{4s^2 \operatorname{Log}_{n-1}^2 s} \right) y(s) = 0.$$

First we use the change of variable and substitution $s = e^{u_1}, y(s) = \sqrt{s} z_1(u_1)$ and if $n > 2$, we continue in the same manner with $u_i = e^{u_{i+1}}, z_i(u_i) = \sqrt{u_i} z_{i+1}(u_{i+1})$ for $i = 2, \ldots, n-1$ to obtain the equation

$$z''_{n-1}(u_{n-1}) + \frac{\tau}{4u_{n-1}^2} z_{n-1}(u_{n-1}) = 0.$$

This equation has a couple of solutions $z_{n-1}(u_{n-1}) = u_{n-1}^{\lambda_{1,2}}$, where $\lambda_{1,2}$ are two (real) zeros of the quadratic equation

$$\lambda^2 - \lambda + \frac{\tau}{4} = \lambda^2 - \lambda + \frac{\mu}{4\mu_p} = 0. \tag{16}$$

Backward transformations result into the couple of solutions of (15)

$$y_{1,2}(t) = \sqrt{\operatorname{Log}_{n-1} t}\,(\log_n t)^{\lambda_{1,2}}$$

and solutions of (14) are

$$\begin{aligned} u_{1,2} = 4\mu_p\, t\frac{y'}{y} &= 4\mu_p\, t\left(\frac{\operatorname{Log}'_{n-1} t}{2\operatorname{Log}_{n-1} t} + \frac{\lambda_{1,2}\log_n' t}{\log_n t}\right) \\ &= 2\mu_p \sum_{i=1}^{n-1} \frac{1}{\operatorname{Log}_i t} + 4\mu_p\lambda_{1,2}\frac{1}{\operatorname{Log}_n t}. \end{aligned} \tag{17}$$

Observe that for $\mu \in (0, \mu_p)$ the zeros $\lambda_{1,2}$ of (16) are in the interval $(0, 1)$. The bigger one $\lambda_1 = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{\mu}{\mu_p}}$ lies in the interval $(\frac{1}{2}, 1)$.

Now let us introduce the function

$$\varphi(t) = \int \frac{u_1^3(s)}{s}\,\mathrm{d}s \tag{18}$$

and the set of functions

$$\mathcal{V} = \{v \in C[T^*, \infty), |v(t) - u_1(t)| \le K\varphi(t)\},$$

where T^* and K will be specified later. Since $\varphi(t) = o(u_1(t))$ as $t \to \infty$ and $u_1(t)$ is positive for large t, there exists T_0 such that $u_1(t) - K\varphi(t) > 0$ for $t > T_0$.

To find a solution of the modified Riccati equation (13), we construct the integral operator

$$\mathcal{F}(v)(t) = \int_t^\infty \left(\sum_{j=1}^{n-1} \frac{\mu_p}{s\operatorname{Log}_j^2 s} + \frac{\mu}{s\operatorname{Log}_n^2 s}\right)\mathrm{d}s + \int_t^\infty \frac{p-1}{s} H(v, \Gamma_p)\,\mathrm{d}s \tag{19}$$

and show that it has a fixed point on the set $\mathcal{V}$.

First we show that the integral $\int^\infty \frac{p-1}{t} H(v, \Gamma_p)\,\mathrm{d}t$ converges. Be T_1 such that the estimate (10) holds. For $v \in \mathcal{V}$ and $t > T_0$ we have $0 < v(t) \le u_1(t) + K\varphi(t)$ and since the function H is increasing in v for $v > 0$, we observe (using Lemma 2.1 and suppressing arguments) that for $t > \max\{T_0, T_1\}$

$$\begin{aligned} \int^\infty \frac{p-1}{t} H(v, \Gamma_p)\,\mathrm{d}t &\le \int^\infty \left|\frac{p-1}{t} H(v, \Gamma_p) - \frac{1}{4\mu_p t} v^2\right| \mathrm{d}t + \int^\infty \frac{1}{4\mu_p t} v^2\,\mathrm{d}t \\ &\le Lb(\varepsilon)\int^\infty \frac{v^3}{t}\,\mathrm{d}t + \int^\infty \frac{v^2}{4\mu_p t}\,\mathrm{d}t \\ &\le Lb(\varepsilon)\int^\infty \frac{(u_1 + K\varphi)^3}{t}\,\mathrm{d}t + \int^\infty \frac{(u_1 + K\varphi)^2}{4\mu_p t}\,\mathrm{d}t < \infty, \end{aligned}$$

since $\varphi(t) = o(u_1(t))$ as $t \to \infty$ and $\int^\infty \frac{u_1^2}{t}\,\mathrm{d}t < \infty$.

Next we show that for suitably chosen constants K and T^* the operator $\mathcal{F}$ maps $\mathcal{V}$ into itself. Using (19) together with (14), Lemma 2.1 and estimates of v, we have for t large enough

$$\begin{aligned}|\mathcal{F}(v)(t)-u_1(t)| &= \left|\int_t^\infty \frac{p-1}{s}H(v,\Gamma_p)\,\mathrm{d}s-\frac{1}{4\mu_p}\int_t^\infty \frac{u_1^2}{s}\,\mathrm{d}s\right| \\ &\le \left|\int_t^\infty \frac{p-1}{s}H(v,\Gamma_p)\,\mathrm{d}s-\frac{1}{4\mu_p}\int_t^\infty \frac{v^2}{s}\,\mathrm{d}s\right|+\frac{1}{4\mu_p}\int_t^\infty \frac{|v^2-u_1^2|}{s}\,\mathrm{d}s \\ &\le Lb(\varepsilon)\int_t^\infty \frac{v^3}{s}\,\mathrm{d}s+\frac{1}{4\mu_p}\int_t^\infty \frac{(u_1+v)|u_1-v|}{s}\,\mathrm{d}s \\ &\le Lb(\varepsilon)\int_t^\infty \frac{(u_1+K\varphi)^3}{s}\,\mathrm{d}s+\frac{1}{4\mu_p}\int_t^\infty \frac{(2u_1+K\varphi)K\varphi}{s}\,\mathrm{d}s.\end{aligned}$$

The first integral satisfies

$$\begin{aligned}\int_t^\infty \frac{(u_1+K\varphi)^3}{s}\,\mathrm{d}s &= \int_t^\infty \frac{u_1^3}{s}\,\mathrm{d}s+\int_t^\infty \frac{3Ku_1^2\varphi+3K^2u_1\varphi^2+K^3\varphi^3}{s}\,\mathrm{d}s \\ &\le \varphi(t)+o(\varphi(t))=\varphi(t)(1+o(1)) \quad \text{as} \quad t\to\infty.\end{aligned}$$

Next, we have

$$\int_t^\infty \frac{K^2\varphi^2}{s}\,\mathrm{d}s=o(\varphi(t)) \quad \text{as} \quad t\to\infty.$$

Finally, we show that

$$\frac{1}{4\mu_p}\int_t^\infty \frac{2u_1K\varphi}{s}\,\mathrm{d}s\le K\lambda_1\varphi(t). \tag{20}$$

Up to this point, first we show that

$$\frac{u_1^2(t)}{t}\le -4\mu_p\lambda_1u_1'(t) \tag{21}$$

for t large enough. Indeed, using (17) and some arrangements, we arrive at the inequality

$$0\le(2\lambda_1-1)\sum_{i=1}^{n-1}\frac{1}{\operatorname{Log}_i^2 t}+2(\lambda_1-1)\sum_{1\le i<j\le n-1}\frac{1}{\operatorname{Log}_i t\operatorname{Log}_j t}+\frac{4\lambda_1(\lambda_1-1)}{\operatorname{Log}_n}\sum_{i=1}^{n-1}\frac{1}{\operatorname{Log}_i t}. \tag{22}$$

Observe that $\lambda_1-1\in(-\frac{1}{2},0)$, $\lambda_1(\lambda_1-1)=-\frac{\mu}{4\mu_p}\in(0,-\frac{1}{4})$ and $2\lambda_1-1\in(0,1)$. Since $\sum_{1\le i<j<n-1}\frac{1}{\operatorname{Log}_i t\operatorname{Log}_j t}=o\left(\sum_{i=1}^{n-1}\frac{1}{\operatorname{Log}_i^2 t}\right)$ and $\sum_{i=1}^{n-1}\frac{1}{\operatorname{Log}_i t\operatorname{Log}_n t}=o\left(\sum_{i=1}^{n-1}\frac{1}{\operatorname{Log}_i^2 t}\right)$, there exists T_2 such that inequality (22) and also (21) is satisfied for $t>T_2$. Now, inequality (21) implies

$$\int_t^\infty \frac{u_1^3(s)}{s}\,\mathrm{d}s\le 2\mu_p\lambda_1\int_t^\infty(-2u_1(s)u_1'(s))\,\mathrm{d}s,$$

which is equivalent to

$$\varphi(t)\le 2\mu_p\lambda_1u_1^2(t).$$

Multiplying by $\frac{u_1}{t}$ leads to the integral inequality

$$\int_t^\infty \frac{u_1(s)\varphi(s)}{s}\,\mathrm{d}s\le 2\mu_p\lambda_1\int_t^\infty \frac{u_1^3(s)}{s}\,\mathrm{d}s$$

and with the use of the definition of φ (18) one can see, that (20) holds for $t > T_2$.

In total, we have

$$|\mathcal{F}(v)(t) - u_1(t)| \leq (Lb(\varepsilon) + K\lambda_1 + o(1))\varphi(t).$$

Let T_3 be so large that the term o(1) is less than or equal to 1 for $t > T_3$. Then for $T^* = \max\{T_0, T_1, T_2, T_3\}$ and for $K \geq \frac{Lb(\varepsilon)+1}{1-\lambda_1}$ (such K exists since $\lambda_1 \in (\frac{1}{2}, 1)$) we obtain

$$|\mathcal{F}(v)(t) - u_1(t)| \leq K\varphi(t).$$

Hence $\mathcal{F}$ maps $\mathcal{V}$ into itself. All the other assumptions of the Schauder-Tichonoff fixed point theorem are satisfied too: $\mathcal{F}(\mathcal{V})$ is bounded and since the derivatives $\mathcal{F}'(v)(t)$ are bounded on compact subintervals of $[T^*, \infty)$, the operator $\mathcal{F}(\mathcal{V})$ is also equicontinuous. Hence the operator $\mathcal{F}(v)(t)$ has a fixed point

$$\mathcal{F}(v)(t) = v(t)$$

on $\mathcal{V}$, for which $v(t) = u_1 + O(\varphi(t))$. The function $v(t)$ now generates a solution of the half-linear Riccati equation (6) $w(t) = \Phi\left(\frac{x'}{x}\right) = h^{-p}(v + G)$, from which one can express a solution of (1) in the form $x(t) = exp\{\int^t \Phi^{-1}(w(s))\,\mathrm{d}s\}$. In more detail, according to (12) and (17),

$$w(t) = \Phi\left(\frac{x'(t)}{x(t)}\right) = 2\mu_p t^{p-1}\left(1 + t\frac{\mathrm{Log}\,'_{n-1}t}{\mathrm{Log}_{\,n-1}t} + 2t\lambda_1\frac{\log'_n t}{\log_n t} + O(\varphi(t))\right).$$

Applying the inverse function Φ^{-1} we have

$$\frac{x'(t)}{x(t)} = \frac{p-1}{p}\frac{1}{t}\left(1 + t\frac{\mathrm{Log}\,'_{n-1}t}{\mathrm{Log}_{\,n-1}t} + 2t\lambda_1\frac{\log'_n t}{\log_n t} + O(\varphi(t))\right)^{q-1}$$

and using the power expansion formula we arrive at

$$\begin{aligned}\frac{x'(t)}{x(t)} &= \frac{p-1}{p}\frac{1}{t}\left(1 + (q-1)t\frac{\mathrm{Log}\,'_{n-1}t}{\mathrm{Log}_{\,n-1}t} + 2t\lambda_1(q-1)\frac{\log'_n t}{\log_n t} + O(\varphi(t))\right)\\ &= \frac{p-1}{p}\frac{1}{t} + \frac{1}{p}\frac{\mathrm{Log}\,'_{n-1}t}{\mathrm{Log}_{\,n-1}t} + \frac{2\lambda_1}{p}\frac{\log'_n t}{\log_n t} + O\left(\frac{\varphi(t)}{t}\right).\end{aligned}$$

Finally, integrating gives (11). □

4. Concluding remarks

1. Within the framework of regularly varying functions we can say that the solution (11) is a regularly varying function of index $\frac{p-1}{p}$ (see the properties of indexes described for example in the Appendix of [14]) and the function $(1 + o(1))$ in its formula is a slowly varying function.
2. The so-called approximate Riccati equation (14) has two linearly independent solutions u_1, u_2 described by (17). To get the solution of (13) and consequently of (1), we used u_1 - the one with the larger zero of (16). The natural question arises whether also u_2 with the smaller zero of (16) could be used to find the second linearly independent solution of (1). This remains as an open problem.

5. Conflict of interest

This work does not have any conflicts of interest.

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