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Abstract

In this work, a laboratory scaled industrial interconnected nonlinear Multi-Input|Multi-Output (MIMO) three-tank system, was modelled to control the liquid levels. Ensuing the tradition in the process industry to apply linear controller to most control processes, a linear control scheme was developed for this system. However, since linear schemes are proximate to actual process models, they may not be adequate, especially for highly nonlinear systems. Therefore, a nonlinear control scheme was also developed and compared with the linear scheme. Specifically, optimal linear and nonlinear controllers were designed. In summary, the results of the two control schemes showed adequate performance. However, the linear controller had more robust control and required lesser computational demand compared to the nonlinear scheme. To enhance the computational demand of the nonlinear scheme, a third-party MATLAB toolbox, Automatic Control and Dynamic Optimization (ACADO) toolbox, that interfaces MATLAB with C++ to speed up computations was also utilised, and its results compared, and tentatively validate the earlier solved nonlinear control scheme.

Keywords: Linear optimal control, nonlinear optimal control, Hamilton-Jacobi equation three-tank-systém, ACADO

1. Introduction

In general, industrial processes are nonlinear, but most control applications are based on linear models, as such controlled via linear control schemes. This is because linear control schemes were first developed, and are now well established, and also cheap to apply [**1,2**]. Identification of a linear model based on process data is relatively easy and linear models provide good results when the plant is operating in the neighbourhood of designed operating conditions. Most nonlinear processes can be accurately described by linear approximate models at specific nominal conditions [**3**] via transfer function, state-space representation, etc. In most control problems, the objective is to keep the process around its steady-state rather than perform frequent changes from one condition to another, and, therefore, a linear model is enough. On the other hand, there are processes for which the nonlinearities are so crucial to the closed-loop stability that a linear model is not sufficient. The Proportional-Integral-Derivative (PID) controllers, which are popular in industrial control [**4,5**], are usually utilised in collaboration with linearised models based on calculated gains at given nominal points. However, depending on the degree of deviation from the nominal points used for linearisation of the model, linear and nonlinear responses can differ significantly [**3**]. And in cases where nominal points may be constantly changing, such linearised model may be grossly inadequate (thus pointing to the need for a nonlinear controller [**6**]), although in these cases gain scheduling can also be incorporated with the PID controller, a simple form of adaptive control [**7,8**]. Overall, the quality of PID controller via linearised models is dependent on the strategy of tuning, and decoupling (i.e., for Multi-Input |Multi-Output, MIMO systems). Tuning strategies such as the Skogestad Internal Model Control (SIMC) rule [**9,10**] have been useful for diverse control scenarios [**11**]. Although, for MIMO, decoupling of input-output interactions must be considered before application of the SIMC tuning rule [**12-14**]. However, optimal controllers, which can be linear or nonlinear, [**15-17**] such as the Linear Quadratic Regulator (LQR), Linear Quadratic Gaussian (LQG), and Model Predictive Controller (MPC, a repetitive optimal controller), decoupling of MIMO system is inherently achieved via calculation of input's optimal from the optimisation of the system [**18,19**]. Although unlike PID controllers, where comprehensive tuning strategies can be applied, optimal controllers are tuned only by changing the input and output weights of the optimisation function, typically a quadratic function. Therefore, in cases when the linear response of the optimal controller is significantly different from the actual process response, these tuning options may be inadequate, and it would be reasonable to develop a nonlinear optimal controller [**6,20**]. However, the controllability and observability of some nonlinear systems may be hard to prove [**21-23**]. Furthermore, nonlinear control theory is still actively being researched, hence there are few methods such as the exact feedback linearisation, backstepping and sliding mode control that can be applied to a generalised nonlinear system [**24**].

Based on the facts drawn from the discussion thus far, which includes: The inherent ability of optimal controllers to decouple inputs and output interactions; The popularity of linear control; and the recommendation of a nonlinear controller for adversely nonlinear systems. Therefore, this paper focuses on applying optimal controllers to a MIMO three-tank system, to compare linear and nonlinear optimal control of a three-tank storage system based on experimentally verified valve constants, and via theoretical laboratory-scale dimensions for the tanks and pumps, utilised by **Kubalcik & Bobal** [**25**]. Aimed at verifying whether for this type of system, it is adequate to use a linear controller, which is much simpler, and cheaper to design, or use a more complex nonlinear control scheme, in the case that the linear control scheme is insufficient for the system, based on the Hamilton-Jacobi equation for linear and nonlinear optimal controllers. Additionally, the Automatic Control and Dynamic Optimization (ACADO) third-party toolbox was also used for the nonlinear control, and its results compared with the programmed nonlinear control scheme. The motivation of this comparative study is because most reports on controlling a three-tank system were based on either linear optimal controller: LQG, LQR, linear MPC, etc., and/or nonlinear MPC controller [**26-31**], with none highlighting the comparative performance of the linear and nonlinear optimal quadratic controllers based on the Hamilton-Jacobi equation. Therefore, the paper's main contribution is the design, proof of applicability, and systematic verification of the efficiency of linear and nonlinear control schemes, based on the Jacobi-Hamilton equation for an interconnected three-tank system. The objectives of this work shall include: Derivation of the nonlinear state model of the system; Static characteristic of the output of the system at varying inputs based on verified experimentally deduced constants; Formulation of nonlinear optimal control design based on the Hamilton-Jacobi equation; Derivation of the linearised state model; Formulation of linear optimal control design based on the Hamilton-Jacobi equation; Simulation and comparison of results from the linear and nonlinear controller.

2. Properties of linear and nonlinear systems

The design of a control system begins with the mathematical formulation and description of the system's dynamics. The control system can be described as linear or nonlinear, depending on the properties of its elements (i.e., inputs, u_i, states, x_i, and outputs, y_i) with time, t. In general, both linear and nonlinear systems exhibit equilibrium state(s), and the stability of the equilibrium state of linear systems depends neither on initial conditions, nor external quantities acting on it, but exclusively on the system's parameters. While the stability of equilibrium state(s) of nonlinear systems is predominantly dependent on the system's parameters, initial conditions, and external quantities. In mathematical terms, a system is linear on the premise that, for every time, t_0 and any two input-output pairs the additive, homogeneous, and superposition property is valid [**32,33**].

Additivity

$$\forall t \geq t_0, \quad y_1(t) + y_2(t) = \begin{cases} x_1(t_0) + x_2(t_0) \\ u_1(t) + u_2(t) \end{cases} \tag{1.1}$$

Homogeneity

$$\forall t \geq t_0, \quad \alpha y_1(t) = \begin{cases} \alpha x_1(t_0) \\ \alpha u_1(t) \end{cases} \tag{1.2}$$

Superposition

$$\forall t \geq t_0, \quad \alpha_1 y_1(t) + \alpha_2 y_2(t) = \begin{cases} \alpha_1 x_1(t_0) + \alpha_2 x_2(t_0) \\ \alpha_1 u_1(t) + \alpha_2 u_2(t) \end{cases} \tag{1.3}$$

The additivity and homogeneity properties combine to yield the superposition property for any real constants α_1 and α_2. A system is termed nonlinear if any of these properties do not hold. Also, an important form of a linear system is the time-invariant property, i.e., a linear function does not depend on time, as illustrated by the differential of the first-order vector, Eq. (**2**) [**32,33**].

$$\begin{aligned} \dot{x} &= f\left[x(t), u(t)\right], \quad \forall t \geq t_0 \\ y &= g\left[x(t), u(t)\right], \quad \forall t \geq t_0 \end{aligned} \tag{2}$$

While the nonlinear time-variant system can either be expressed in continuous, Eq. (**3.1**) or discretised, Eq. (**3.2**) form.

$$\begin{aligned} \dot{x} &= f\left[t, x(t), u(t)\right], \quad \forall t \geq t_0 \\ y &= g\left[t, x(t), u(t)\right], \quad \forall t \geq t_0 \end{aligned} \tag{3.1}$$

$$\begin{aligned} x(k+1) &= f\left[k, x(k), u(k)\right], \quad k = 0, 1, 2, \ldots, N-1 \\ y(k) &= g\left[k, x(k), u(k)\right], \quad k = 0, 1, 2, \ldots, N-1 \end{aligned} \tag{3.2}$$

Where $x\ (t)$ is a n-dimensional column state vector, $x \in \mathcal{R}$, $u\ (t)$ is m-dimensional column input vector, $u \in \mathcal{R}^m$ and $y\ (t)$ is a p-dimensional column output vector, $y \in \mathcal{R}^p$. Also, f and g are vector functions of the state and input respectively, t is a scalar variable, $t \in \mathcal{R}_+$, and k is the discretised point of t and t_0 is the initial time.

3. Process description and model development

3.1. Process description

The system of consideration is a laboratory-scaled interconnected tank, comparable to those used for the storage of liquid products in food processing, chemical, and petrochemical plants. Industrial storage tanks are usually interconnected, for economic and logistical reasons. It minimises pumping energy and ensures adequate distribution of products. A three-tank system, Tank-1, Tank-2, and Tank-3 considered in this work, **Fig. 1** are connected in series via cylindrical pipes fitted with flow valves that control flows: from Tank-1 to Tank-2, q_1 via a flow constant k_1; Tank-2 to Tank-3, q_2 via a flow constant k_2; Tank-1 to the surrounding, q_3 via a flow constant k_3; Tank-3 to surrounding, q_4 via a flow constant k_4. The Tank-2 and Tank-3 are connected to pumps with respective flow rates, Q_1 and Q_2, that channels fluid from a reservoir. The pump flow rates together with the flow valves can be used to change the liquid heights, h_1, h_2, and h_3 of the respective tanks. However, it will be assumed that the flow valves are always open, the flow rate of the pumps Q_1 and Q_2 are manipulable and since Tank-2 does not receive fluids directly, its height, h_2 is considered directly uncontrollable. Therefore, the controllable outputs for this system would be h_1, and h_3, manipulable via Q_1 and Q . Based on this fact, it can be inferred that coupling (i.e., the interaction between the controlled and manipulable variables) will occur since each of these flow rates will simultaneously affect both heights.

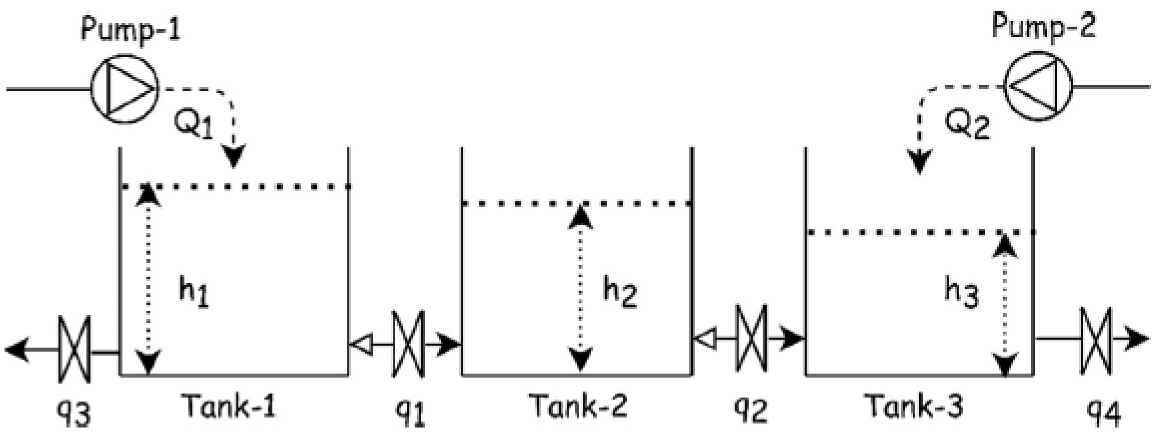

Fig. 1. Diagrammatic description of the three-tank system.

3.2. Model development

Having described the three-tank system, the law of conservation of matter can be applied to the system, following the direction (i.e. bold arrowheads) of the input and output vectors on each tank, **Fig. 1**, to derive the set of volumetric dynamics for Tank-1, Tank-2, and Tank-3 respectively, Eqs. (**4.1**)-(**4.3**).

$$\frac{dV_1}{dt} = Q_1 - q_1 - q_3 \tag{4.1}$$

$$\frac{dV_2}{dt} = q_1 - q_2 \tag{4.2}$$

$$\frac{dV_3}{dt} = Q_2 + q_2 - q_4 \tag{4.3}$$

Furthermore, assuming Torricelli's law applies to liquid flow rates through the valves i.e., the flow rate is dependent on the liquid level, valve position, and constant, as approximated by Eqs. (**5.1**)-(**5.4**). Based on Eqs. (**5.1**) and (**5.2**), and as indicated by the two arrowheads in **Fig. 1**. The direction of q_1 and q_2 is dependent on h_1 and h_2.

$$q_1 = sign(h_1 - h_2)k_1\sqrt{|h_1 - h_2|} \tag{5.1}$$

$$q_2 = sign(h_2 - h_3)k_2\sqrt{|h_2 - h_3|} \tag{5.2}$$

$$q_3 = k_3\sqrt{h_1} \tag{5.3}$$

$$q_4 = k_4\sqrt{h_3} \tag{5.4}$$

Considering equal cross-sectional areas of Tank-1, Tank-2 and Tank-3 i.e. A, so their corresponding liquid volumes are $V_1 = Ah_1$, $V_2 = Ah_2$, and $V_3 = Ah_3$, and by substituting Eqs. (**5.1**)-(**5.4**) into Eqs. (**4.1**)-(**4.3**) and rearranging, yields Eqs. (**6.1**)-(**6.3**).

$$\frac{dh_1}{dt} = 1/A\left(Q_1 - k_1 sign(h_1 - h_2)\sqrt{|h_1 - h_2|} - k_3\sqrt{h_1}\right) \tag{6.1}$$

$$\frac{dh_2}{dt} = 1/A\left(k_1 sign(h_1 - h_2)\sqrt{|h_1 - h_2|} - k_2 sign(h_2 - h_3)\sqrt{h_2 - h_3}\right) \tag{6.2}$$

$$\frac{dh_3}{dt} = 1/A\left(Q_2 + k_2 sign(h_2 - h_3)\sqrt{h_2 - h_3} - k_4\sqrt{h_3}\right) \tag{6.3}$$

In summary, Q_1 and Q_2 constitutes two input variables, u_i and the liquid levels, h_1 and h_3 constitutes two output variables, y_i, therefore the process is a MIMO system. Overall, the liquid level, h_1, h_2, and h_3, constitutes three state variables, xi of the system. Furthermore, because state variables are expressed as square root functions, the system is termed a nonlinear MIMO system.

The nonlinearity of the system can also be illustrated by the system's static characteristics, **Fig. 2** via simulation of varied inputs, using the experimentally deduced and verified valve constants given in **Table 1**. Before developing **Figs. 2**(b)-**2**(c), a twodimensional plot for the state variables was generated, **Fig. 2**(a), to determine the adequate simulation time required to deduce fully developed static characteristics of the system.

Table 1 Experimentally deduced parameters for the three-tank systém

Parameters	Values and units (1 cm = 0.01 m)
Tank cross-section area, A	7 cm^2
q_1 valve coefficient	5 $cm^{5/2}$ s^{-1}
q_2 valve coefficient	4 $cm^{5/2}$ s^{-1}
q_3 valve coefficient	3 $cm^{5/2}$ s^{-1}
q_4 valve coefficient	2 $cm^{5/2}$ s^{-1}

4. Theory and methodologies

4.1. Nonlinear quadratic optimal control formulation

Optimal control problems can be solved by direct or indirect methods that require the solution of a complex set of equations such as the Euler-Lagrange differential and Kuhn-Tucker algebraic equation, implemented via numerical iterations with the aid of an initial guess solution [**34**]. Consider the continuous dynamic problem given by Eq. (**3.1**), with n differential states, $x\ (t)$, and m control functions, $u\ (t)$, to be solved conjugatively with the performance index, Eq. (**7.1**), based on the generalised calculus of variation, which can also be expressed as a quadratic index, Eq. (**7.2**), and can also be further reduced to Eq. (**7.3**) via a zero-terminal index (i.e. $\varphi = 0$) assumption. These indices are designed to be minimised to deduce a sequence of $u\,(t)$. The actual implementation of the solution to this problem is equivalent to solving the associated Hamilton-Jacobi equation [**15,34-37**].

$$\min_u J = \varphi\left(x\left(t_f\right), t_f\right) + \int_{t_0}^{t_f} L\,(x, u, t)\,dt \tag{7.1}$$

$$\min_u J = \frac{1}{2}x^T\left(t_f\right) S\left(x\left(t_f\right), t_f\right) x\left(t_f\right) + \frac{1}{2}\int_{t_0}^{t_f}\left[x^T W\,(x, t)\,x + u^T R\,(x, u, t)\,u\right] dt \tag{7.2}$$

$$\min_u J = \frac{1}{2}\int_{t_0}^{t_f}\left[x^T W\,(x, t)\,x + u^T R\,(x, u, t)\,u\right] dt \tag{7.3}$$

Where $S \in \mathcal{R}^{n\,x\,n}$ is the state weighing matrix of the zero-terminal index, $W \in \mathcal{R}^{n\,x\,n}$ is the symmetric and nonnegative matrix of the states weighting matrix, and $R \in \mathcal{R}^{n\,x\,m}$ is a symmetric positive input weighting matrix for all $x\,(t)$. In specific terms, the nonlinear dynamic equation for this work, Eq. (**6**) in relation to Eq. (**3.1**) can be expressed as given by Eq. (**8.1**). Noting also the input is as implied by Eq. (**8.2**).

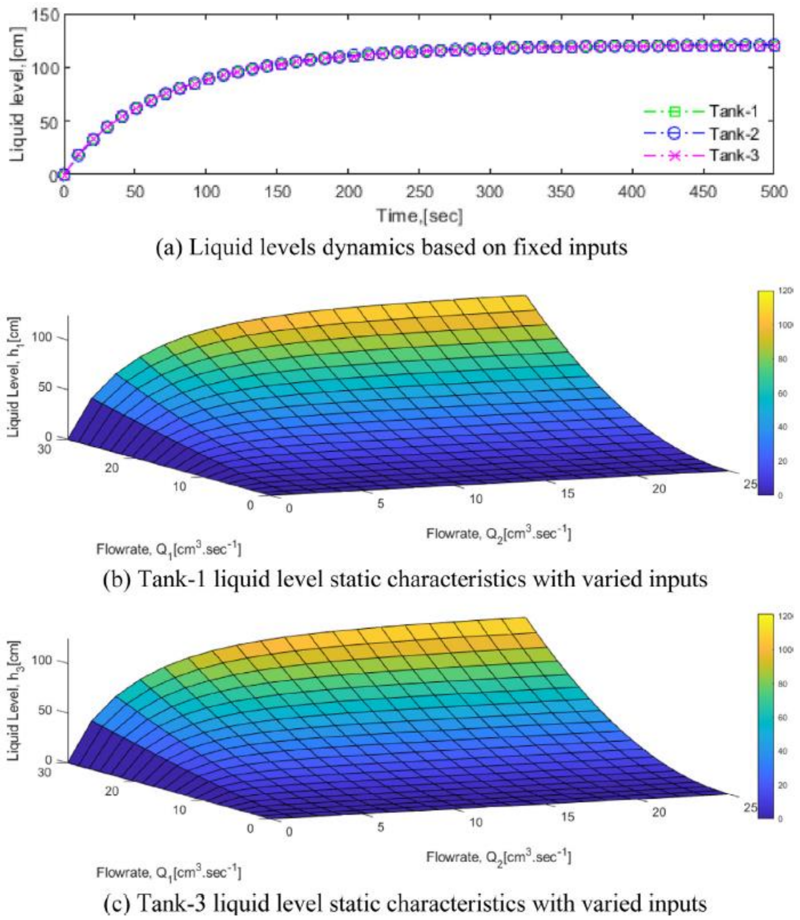

Fig. 2. Illustration of tanks liquid level changes with time and inputs.

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} \Longrightarrow \begin{bmatrix} \dot{h}_1 \\ \dot{h}_2 \\ \dot{h}_3 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1/A\left(Q_1 - k_1 sign(h_1 - h_2)\sqrt{|h_1 - h_2|} - k_3\sqrt{h_1}\right) \\ 1/A\left(k_1 sign(h_1 - h_2)\sqrt{|h_1 - h_2|} - k_2 sign(h_2 - h_3)\sqrt{h_2 - h_3}\right) \\ 1/A\left(Q_2 + k_2 sign(h_2 - h_3)\sqrt{h_2 - h_3} - k_4\sqrt{h_3}\right) \end{bmatrix}}_{f[t, x(t), u(t)]} \tag{8.1}$$

$$y(t) = \begin{bmatrix} x(t)_1 \\ x(t)_3 \end{bmatrix} \Longrightarrow \underbrace{\begin{bmatrix} h_1 \\ h_3 \end{bmatrix}}_{g[t, x(t), u(t)]}$$

$$u(t) = \begin{bmatrix} u(t)_1 \\ u(t)_2 \end{bmatrix} \Longrightarrow \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \tag{8.2}$$

The traditional zero-terminal quadratic performance index, Eq. (**7.3**) is utilised for this nonlinear optimal control. However, optimal control problems are usually implemented in discretised form i.e., via Eq. (**3.2**) as illustrated by Eq. (**9**). Where the time scale of Eq. (**8.1**) is divided into discrete points, $k \in [1\ N]$, and $y(k)_{ref}$ is the reference values of outputs. Note that in Eq. (**9**), the input, $u(k)$ is considered an inequality constraint, as the pump's flow rate is usually limited by the manufacturers' specification.

$$\min_u J = \frac{1}{2}\sum_{k=1}^{N}(y\,(k)_{ref} - y(k))^T W(y(k)_{ref} - y(k)) + u\,(k)^T\, Ru(k)$$

Subject to: (9)

$$x\,(k+1) = f\,[k, x\,(k)\,, u\,(k)]$$

$$y\,(k) = g\,[k, x\,(k)\,, u\,(k)]$$

$$\begin{bmatrix} Q_{1,min} \\ Q_{2,min} \end{bmatrix} \Longleftarrow \begin{bmatrix} u(k)_{1,min} \\ u(k)_{2,min} \end{bmatrix} \le u(k) \le \begin{bmatrix} u(k)_{1,max} \\ u(k)_{2,max} \end{bmatrix} \Longrightarrow \begin{bmatrix} Q_{1,max} \\ Q_{2,max} \end{bmatrix}$$

4.2. Linear quadratic optimal control formulation

Following the analogy of Eq. (**9**) and utilising a discretised system in the form of Eq. (**3.2**), obtained after Taylor's series or Jacobian linearisation of Eq. (**8.1**). The linear optimal control problem for this case study is structured as given by Eq. (**10**). Where Ak, and Bk are matrices of the coefficients of discretised states, and inputs. Note that, unlike the nonlinear optimal controller, the linear optimal controller can only perform Real-time optimisation (RTO), when the process is in a steady-state [**38**].

$$\min_u J = \frac{1}{2}\sum_{k=1}^{N}(y\,(k)_{ref} - y(k))^T W(y(k)_{ref} - y(k)) + u\,(k)^T Ru(k)$$

$$\text{Subject to:} \quad (10)$$

$$x\,(k+1) = A_k x\,(k) + B_k u\,(k)$$

$$y\,(k) = Cx\,(k)$$

$$\begin{bmatrix} Q_{1,min} \\ Q_{2,min} \end{bmatrix} \Longleftarrow \begin{bmatrix} u(k)_{1,min} \\ u(k)_{2,min} \end{bmatrix} \le u(k) \le \begin{bmatrix} u(k)_{1,max} \\ u(k)_{2,max} \end{bmatrix} \Longrightarrow \begin{bmatrix} Q_{1,max} \\ Q_{2,max} \end{bmatrix}$$

The linearisation of the continuous nonlinear model, Eq. (**8.1**), is executed at calculated nominal and equilibrium values of the inputs, $\bar{u}$ and states, $\bar{x}$ as given via the Jacobian linearisation method, Eq. (**11**). Note that the coefficients of the matrices A_t and B_t for the linearisation step of $\dot{x} = f\,[t, x\,(t), u\,(t)]$ to $\dot{x} = A_t\,(t) + B_t u\,(t)$ are different from A_k and B_k. Where $A_k = e^{A_t T_s}$ and $B_k = A_t^{-1}(A_k - IB_t)$, with $t = kT_s$, T_s the sampling time based on the zero-order hold estimation [**39,40**]. While C is the matrix of zeros and ones that defines the outputs from the states, as such same for the linearisation and discretisation steps.

$$\dot{x} \Longrightarrow \begin{bmatrix} \dot{h}_1 \\ \dot{h}_2 \\ \dot{h}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \left(\frac{df}{dh_1}\right)_{\substack{h=\bar{h}\\ Q=\bar{Q}}} & \left(\frac{df}{dh_2}\right)_{\substack{h=\bar{h}\\ Q=\bar{Q}}} & \left(\frac{df}{dh_3}\right)_{\substack{h=\bar{h}\\ Q=\bar{Q}}} \\ \left(\frac{df}{dh_1}\right)_{\substack{h=\bar{h}\\ Q=\bar{Q}}} & \left(\frac{df}{dh_2}\right)_{\substack{h=\bar{h}\\ Q=\bar{Q}}} & \left(\frac{df}{dh_3}\right)_{\substack{h=\bar{h}\\ Q=\bar{Q}}} \\ \left(\frac{df}{dh_1}\right)_{\substack{h=\bar{h}\\ Q=\bar{Q}}} & \left(\frac{df}{dh_2}\right)_{\substack{h=\bar{h}\\ Q=\bar{Q}}} & \left(\frac{df}{dh_3}\right)_{\substack{h=\bar{h}\\ Q=\bar{Q}}} \end{bmatrix}}_{A_t} \underbrace{\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}}_{x(t)}$$

$$+ \underbrace{\begin{bmatrix} \left(\frac{df}{dQ_1}\right)_{\substack{h=\bar{h}\\ Q=\bar{Q}}} & \left(\frac{df}{dQ_2}\right)_{\substack{h=\bar{h}\\ Q=\bar{Q}}} \\ \left(\frac{df}{dQ_1}\right)_{\substack{h=\bar{h}\\ Q=\bar{Q}}} & \left(\frac{df}{dQ_2}\right)_{\substack{h=\bar{h}\\ Q=\bar{Q}}} \end{bmatrix}}_{B_t} \underbrace{\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}}_{u(t)} \quad (11)$$

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}}_{x(t)}$$

Given the nominal values of inputs,_$\bar{Q}$ i.e., $\bar{Q}_1$ and $\bar{Q}_2$, the equilibrium values of the states $\bar{h}$ i.e., $\bar{h}_1$, $\bar{h}_2$ and $\bar{h}_3$ are deduced at steady-state condition, i.e., from Eq. (**8.1**) at $\dot{x}$; $\dot{h} = 0$, as given by Eq. (**12**).

$$\dot{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/A\left(\overline{Q}_1 - k_1 sign(\overline{h}_1 - \overline{h}_2)\sqrt{|\overline{h}_1 - \overline{h}_2|} - k_3\sqrt{\overline{h}_1}\right) \\ 1/A\left(k_1 sign(\overline{h}_1 - \overline{h}_2)\sqrt{|\overline{h}_1 - \overline{h}_2|} - k_2 sign(\overline{h}_2 - \overline{h}_3)\sqrt{\overline{h}_2 - \overline{h}_3}\right) \\ 1/A\left(\overline{Q}_2 + k_2 sign(\overline{h}_2 - \overline{h}_3)\sqrt{\overline{h}_2 - \overline{h}_3} - k_4\sqrt{\overline{h}_3}\right) \end{bmatrix} \quad (12)$$

4.3. Methodologies

Having formulated the linear and nonlinear optimal control problem, the solution is implemented via MATLAB based on the optimal control algorithm described by Algorithm 1. The initial guesses for $u\ (k)$ $\Rightarrow Q_1$ & Q_2 and $x\ (k) \Rightarrow h_1$, h_2, & h_3 are the same as the initial values given in **Table 2**.

Table 2 Proposed simulation parameters

Parameters	Values (unit), (1 cm = 0.01 m)
Output weighing factors, W_1 & W_2	1.0 & 1.0
Input weighing factors, R_1 & R_2	0.01 & 0.01
Sampling time, T_s	0.4 s
Pump 1 minimum flow rate, $Q_{1,min}$	0 cm^3 s^{-1}
Pump 1 maximum flow rate, $Q_{1,max}$	20 cm^3 s^{-1}
Pump 2 minimum flow rate, $Q_{2,min}$	0 cm^3 s^{-1}
Pump 2 maximum flow rate, $Q_{2,max}$	10 cm^3 s^{-1}
Pumps (Q_1 & Q_2) initial flow rate	20 & 10 cm^3 s^{-1}
Tanks (h_1, h_2, & h_3) initial height	0, 0 & 0 cm

Algorithm 1. Optimal control algorithm

1. Starting with an initial guess for $u(k)$ and $x(k)$
2. For $k = 1, 2, \ldots$ Iterate until convergence
3. Computation route for a step ahead of the state, $x(k+1)$ and corresponding $y(k)$ based on previous $u(k)$ and $x(k)$
4. Execute optimisation routine based on the objective function and with $x(k+1)$ to deduce $u(k+1)$
5. If $\|u(k+1) - u(k)\| < \varepsilon$, then End ($u(k+1)$ is the optimal input), else continue iteration

Where ε is an arbitrarily small positive number.

In the algorithm the step 3 and step 4 for linear optimal control was implemented using the Kronecker products of states-space matrices, and ''quadprog'' function, while for nonlinear optimal control the Runge-Kutta algorithm and ''fmincon'' (using the ''sqp'' option) are respectively utilised in this study. In addition, a third-party control toolkit, Automatic Control and Dynamic Optimization (ACADO) (via the ''minimizeLSQ'' route) [**41**] was utilised for solving the nonlinear optimal control.

The tuning parameters, i.e., output, W and input, R weighting matrices are diagonal matrices as expressed by Eq. (**13**), where W_1 & W_2 and R_1 & R_2 are respectively weighing factors for outputs and inputs. Furthermore, typical values of parameters used for the simulation are given in **Table 2**.

$$W = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} \& R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \quad (13)$$

5. Results and discussion

Starting withfinear optimal control, the nominal inputs, $\overline{Q}_1 = 20$ cm^3 s^{-1} & $\overline{Q}_2$ = 10 cm^3 s^{-1} and calculated equilibrium values for $\bar{h}_1 = 36.1579$ cm, $\bar{h}_2 = 36.0041$ cm and $\bar{h}_3$ = 35.7639 cm based on Eq. (**12**) were used to develop the controller. The resulting matrices of the continuous, A_t & B_t and discretised, A_k & B_k linear models are given by Eq. (**14**).

$$A_t = \begin{bmatrix} -0.9464 & 0.9108 & 0 \\ 0.9108 & -1.4940 & 0.5829 \\ 0 & 0.5829 & -0.6068 \end{bmatrix} \& B_t = \begin{bmatrix} 0.1429 & 0 \\ 0 & 0 \\ 0 & 0.1429 \end{bmatrix} \quad (14.1)$$

$$A_k = \begin{bmatrix} 0.7278 & 0.2312 & 0.0288 \\ 0.2312 & 0.6073 & 0.1587 \\ 0.0283 & 0.1587 & 0.8038 \end{bmatrix} \& B_k = \begin{bmatrix} 0.0485 & 0.0006 \\ 0.0077 & 0.0051 \\ 0.0006 & 0.0511 \end{bmatrix} \quad (14.2$$

Upon discretisation, the linear and nonlinear optimal control schemes were used to simulate the control of the_system output, $y(k) = [h_1\ h_3]^T$ to references, $y\ (k)_{ref} = [\bar{h}_1\ \bar{h}_3]^T$ for unconstrained conditions, **Fig. 3**. The result indicates both control schemes attained the references adequately, quickly, and smoothly, however, the nonlinear scheme experienced initial noticeable overshoot and undershoot (an indication of potential robustness problems) from the references due to initial over reactiveness of the inputs, and long control duration. Although for systems where such a robustness problem is likely to adversely affect the quality of the output, it would be necessary to apply input or output constraints. Also, slight oscillations were observed for the nonlinear controller throughout the control duration.

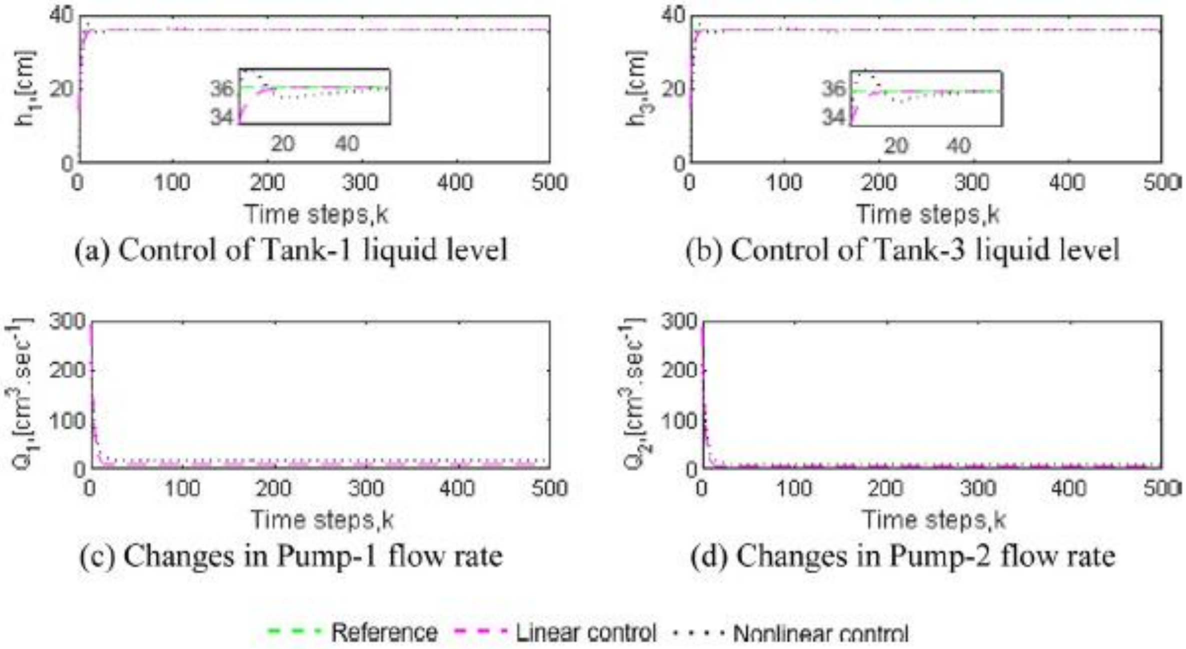

Fig. 3. Changes in tanks heights and pumps, flow rates due to unconstrained controller's action to a static reference.

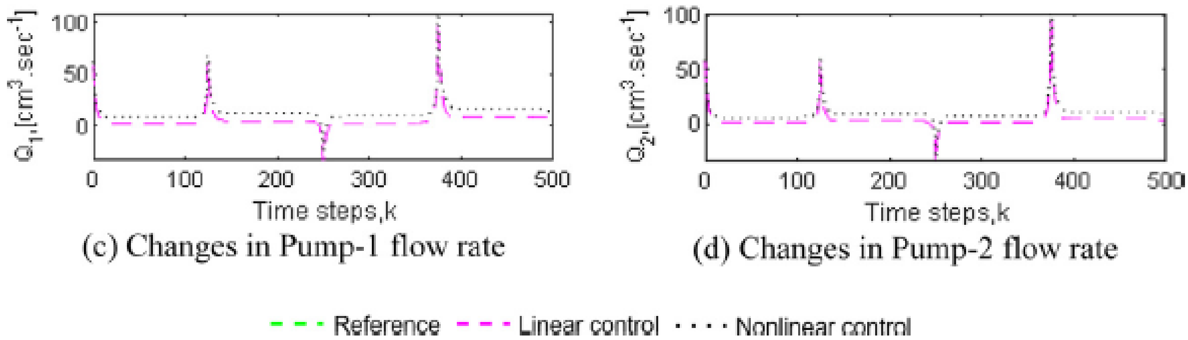

Fig. 4. Changes in tanks heights and pumps, flow rates due to unconstrained controller's action to periodic and successively changing references.

Furthermore, the reliability (i.e., ability to quickly attain and maintain periodic references) of these two control schemes was investigated by subjecting the controllers to periodic and successively changing references, $y\ (k)_{ref}$ of 20%, 50%, 30%, and 80% of the equilibrium value of the outputs, $\bar{h}_1$ and $\bar{h}_2$ at every quartile of the total time of the simulation. The result of the simulation, **Fig. 4**, and their performance is like the result in **Fig. 3**.

In practice, the inputs are likely to have a limited range, as such results in **Figs. 3** and **4** may not be applicable. Therefore, using the earlier specified $y\ (k)_{ref}$, a case study of constrained inputs as illustrated in Eqs. (**9**) and (**10**) and with the values given in **Table 2**, is considered. i.e., with maximum limits set at the given input nominals. The result, **Fig. 5**(a) & **5**(b) indicates both control schemes attained the references adequately and smoothly, but slowly due to the added constraints. However, the linear control was faster than the nonlinear control, this is due to a wider allowable limit for manipulation of inputs for the linear controller. This is because, unlike the nonlinear system whose nominal inputs remain the same, on linearisation to deduce the linear system the nominal value of the inputs shifts to about 47.26% of $\overline{Q}_1$ and 55.31% of $\overline{Q}_2$, **Fig. 5**(c) & **5**(d). This shift can be verified by recomputing the nominal value of inputs, $\overline{Q}_1$ and $\overline{Q}_2$ of_the Jaco_bian linearised model, Eqs. (**11**) and (**14.1**) using $h_1 = \bar{h}_1$, $h_2 = \bar{h}_2$, $h_3 = \bar{h}_3$ and at $\dot{x} = 0$ as given in Eq. (**15**).

$$\dot{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.9464\bar{h}_1 + 0.9108\bar{h}_2 + 0 + 0.1429\overline{Q}_1 + 0 \\ 0.9108\bar{h}_1 - 1.4940\bar{h}_2 + 0.5829\bar{h}_3 + 0 + 0 \\ 0 + 0.5829\bar{h}_2 - 0.6068\bar{h}_3 + 0 + 0.1429\overline{Q}_2 \end{bmatrix} \quad (15)$$

As earlier analysed, the reliability of these two constrained control schemes were also investigated by subjecting the controllers to the same earlier specified periodic and successively changing references, and the result, **Fig. 6**, was like that of **Fig. 5**. Although, while both controllers were able to quickly attain and maintain the lower periodic references, as the references get higher, approaching the equilibrium output values, the responsiveness of the nonlinear controller becomes slower.

Therefore, on the premise of the discussion thus far, it may be suggested that for the nonlinear scheme to have the same performance as the linear system, the upper bound of the inputs constraints should be increased to the same proportion as the shift in the nominals of inputs in the linear control scheme, **Figs. 5**(c) & **5**(d), i.e., $Q_{1,\max}$ from 20 to 42 cm^3 s^{-1} and $Q_{2,\max}$ from 10 to 18 cm s^{-1} in this case. The result, **Fig. 7** proves this premise valid, the nonlinear control has been just as responsive as the linear control scheme, except for slight oscillations in attaining the second and last periodic reference level for Tank-1.

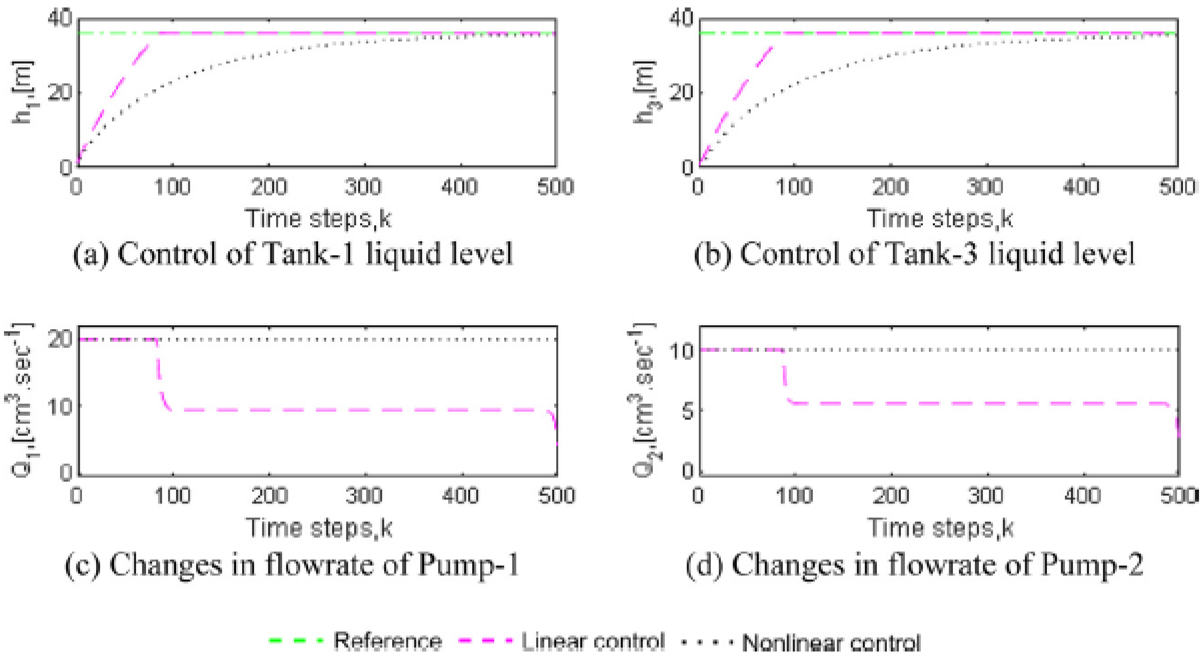

Fig. 5. Changes in tanks heights and pumps, flow rates due to constrained controller's action to static references.

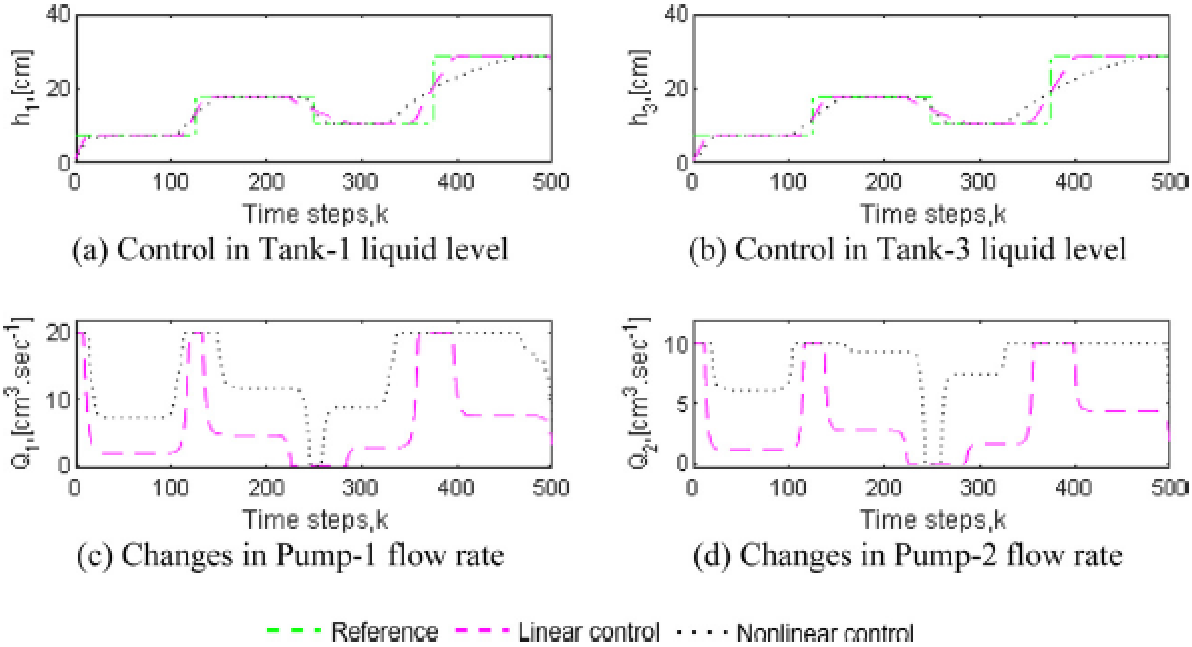

Fig. 6. Changes in tanks heights and pumps, flow rates due to constrained controller's action to periodic and successively changing references.

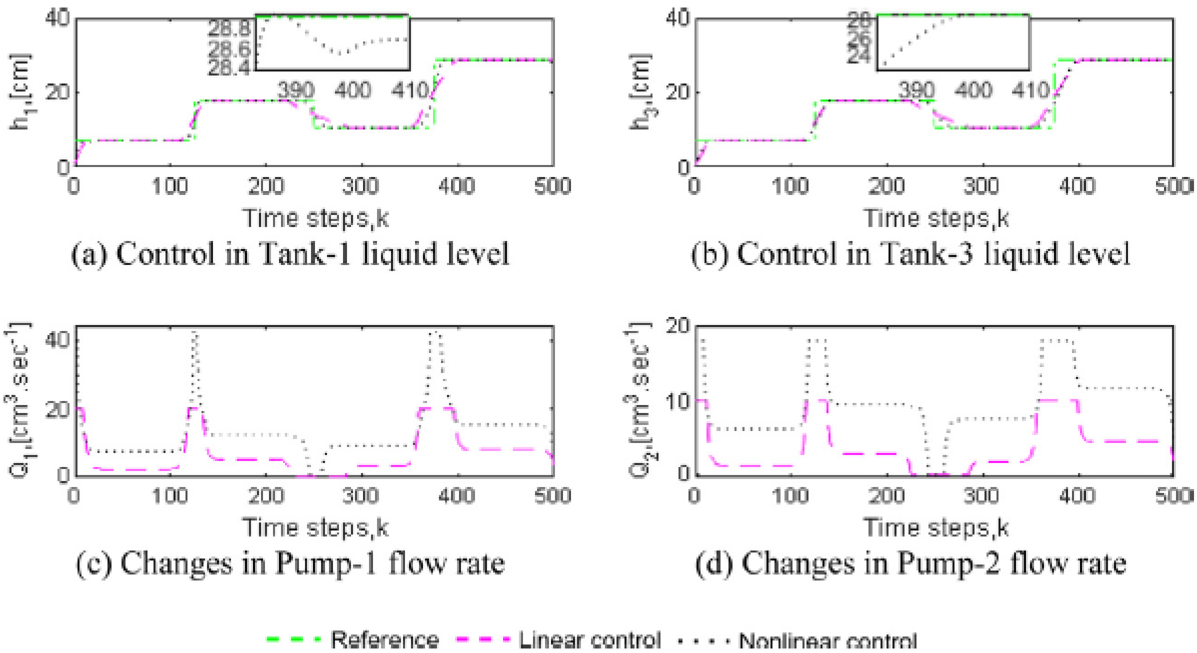

Fig. 7. Changes in tanks heights and pumps, flow rates due to constrained controller's action to periodic and successively changing references.

In addition to the potential robustness problems, it was observed that the nonlinear control scheme is more computationally demanding and as such required a longer time for simulation (i.e., clock time). Specifically, as an illustration of this fact, the results shown in **Fig. 6** took about 4 min to simulate the linear control scheme and 23 min for the nonlinear control scheme (via a 3.59 GHz 4-core AMD Ryzen 3 3100 processor computer coupled with 16.0 GB of RAM).

To improve the computational speed, and tentatively compare and validate the nonlinear control scheme used thus far, the ACADO toolbox, which interfaces MATLAB with C++ via a C++ compiler was utilised, using the preceding conditions of the nonlinear model in **Fig. 7** i.e., a periodic and successively changing references and constrained at its upper bound (i.e., $Q_{1,max} \approx 42$ cm^3 s^{-1} and $Q_{2,\max} \approx 18$ cm^3 s^{-1}). The resulting simulation is illustrated in **Fig. 8**. The computation speed for ACADO in this case study was 4 times faster, and the results are quite similar, but unlike the earlier implemented nonlinear control scheme, results from ACADO showed no sign of oscillation in attaining the second and last periodic references, **Figs. 8**(a) and **8**(b). Also, there is a noticeable difference in inputs changes between the two results, **Figs. 8**(c) and **8**(d) but not as obvious as between the linear and earlier nonlinear control schemes, **Figs. 7**(c) and **7**(d). These differences are obviously due to the difference in the implementation algorithm between the two approaches.

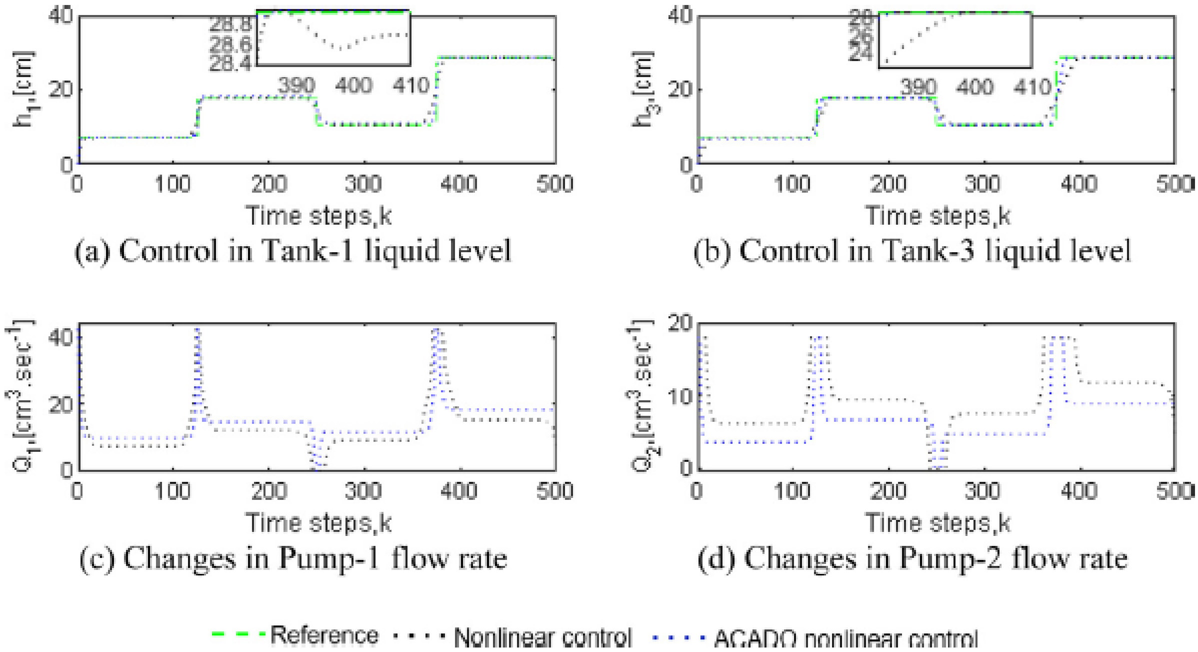

Fig. 8. Changes in tanks heights and pumps, flow rates due to constrained controller's action to periodic and successively changing references.

The simulation results for linear and nonlinear quadratic optimal control scheme discussed thus far is quite like those reported in literature. Parikh et al. [**30**] compared LQG control and nonlinear MPC for the same three-tank system, the result indicated a robustness problem for the nonlinear MPC scheme (i.e., a repetitive optimal controller) as also observed in this work. However, as opposed to this work, Parikh et al. [**30**] reported a robustness problem with the linear controller. This difference could be attributed to the difference between the linear quadratic optimal and LQG control model, and/or the difference in simulation time. Although Lengyel et al. [**31**] report for level control of a cascaded multi-tank system via LQR model indicated no such robustness problem, however, it should be noted the simulation time was shorter in comparison to this work and that of Parikh et al. [**30**]'s result. Furthermore, regarding the computational demand, Parikh et al. [30] also indicated that the linear control scheme was faster in attaining the reference than the nonlinear control scheme, in conformity to the result of this work.

6. Conclusion

In summary, the main aim of this paper, which is to evaluate the advantages and disadvantages of two different approaches to controlling the liquid level of a nonlinear MIMO three interconnected tanks system, was successfully achieved. The first method is based on the design of a linear controller, while the second one is based on the design of a nonlinear controller. Although simulation results of both approaches showed good control quality as the system was stabilised and asymptotic tracking of the reference signals was also achieved. However, observable and comparable qualities of both control schemes were also noticeable. The linear control scheme showed no robustness problem and was less computational demanding in comparison to the nonlinear control schemes. Unlike the nonlinear schemes, the linear scheme is not the true reflection of a practical control system, because real systems are likely to experience oscillations as such the precondition for a linear controller, i.e., an assumption of the steady-state condition of practical systems is not frequently achievable and requires a significant amount of time in the industry — especially for large systems. This assumption limits the application of the linear scheme, and as such, a nonlinear controller would be a better choice for more accurate proximity to practical systems. Furthermore, comparison of the nonlinear control scheme to ACADO (i.e., a third-party nonlinear control toolbox), showed the problems highlighted so far have not been observed in the implementation with ACADO. In short, ACADO's performance was more like a linear control scheme in terms of computational speed and robustness. Conclusively, both controller schemes were able to perform the primary objective of controlling the liquid level of the system. However, regarding the choice scheme to apply, a trade-off between accurate proximity to the system, and ease or cost of implementation must be decided by stakeholders. Having shown the proposed linear and nonlinear quadratic optimal control schemes are adequate to control the liquid level of a nonlinear MIMO three interconnected tanks system. Either of these control schemes based on the choice of highlighted trade-off can be used as an alternative to the popular MPC scheme (a repetitive optimal controller) to control the liquid level of nonlinear process systems such as industrial tank farms for chemicals, petroleum, and food products. Furthermore, future research will focus on how to reduce the computational speed and enhance the robustness of the proposed nonlinear quadratic optimal controller by improving the implementation of its algorithm, within or exceeding the performance of ACADO toolbox.

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