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Perturbed generalized half-linear Riemann–Weber equation – further oscillation results

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Abstract. We establish new oscillation and nonoscillation criteria for the perturbed generalized Riemann–Weber half-linear equation with critical coefficients

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \sum_{j=1}^{n} \frac{\mu_p}{t^p \mathrm{Log}_j^2 t} + \tilde{c}(t)\right)\Phi(x) = 0$$

in terms of the expression

$$\frac{1}{\log_{n+1} t}\int^t \tilde{c}(s)s^{p-1}\,\mathrm{Log}_{\,n} s \log_{n+1}^2 s \,\mathrm{d}s.$$

The obtained criteria complement results of [O. Došlý, *Electron. J. Qual. Theory Differ. Equ., Proc. 10'th Coll. Qualitative Theory of Diff. Equ.* **2016**, No. 10, 1–14].

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1 Introduction

Consider the half-linear differential equation of the form

$$L[x] := (r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \qquad \Phi(x) = |x|^{p-1}\operatorname{sgn} x, \qquad p > 1, \tag{1.1}$$

where r, c are continuous functions, $r(t) > 0$ and $t \in [T, \infty)$ for some $T \in \mathbb{R}$. The terminology *half-linear* comes from the fact that the solution space of (1.1) is homogenous, but generally not additive for $p \neq 2$. In the special case $p = 2$ this equation reduces to the linear Sturm–Liouville differential equation

$$(r(t)x')' + c(t)x = 0. \tag{1.2}$$

In this paper we deal with oscillatory properties of equations of the form (1.1). It is well known that the classical linear Sturmian theory of (1.2) can be naturally extended also to (1.1), see [8].

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In particular, (1.1) is called *oscillatory* if all of its solutions are oscillatory, i.e., it has infinitely many zeros tending to infinity. In the opposite case all solutions of (1.1) are nonoscillatory, i.e., they are eventually positive or negative and (1.1) is said to be *nonoscillatory*. Let us emphasize that oscillatory and nonoscillatory solutions of (1.1) cannot coexist.

If we suppose that (1.1) is nonoscillatory, one can study the influence of the perturbation $\tilde{c}$ on the oscillatory behavior of the equation of the form

$$(r(t)(\Phi(x'))' + (c(t) + \tilde{c}(t))\Phi(x) = 0. \tag{1.3}$$

The concrete (non)oscillation criteria measure the positiveness of the function $\tilde{c}$ (generally of arbitrary sign). If $\tilde{c}$ is "sufficiently positive" then the perturbed equation (1.3) becomes oscillatory, if $\tilde{c}$ is negative or "not too much positive", then (1.3) remains nonoscillatory. This approach is sometimes referred to as the perturbation principle and leads, e.g., to the Hille–Nehari type (non)oscillation criteria for (1.3) which compare limits inferior and superior of certain integral expressions with concrete constants. These integral expressions are usually either of the form

$$\int_T^t R^{-1}(s)\,\mathrm{d}s \int_t^\infty \tilde{c}(s)h^p(s)\,\mathrm{d}s \quad \text{if} \quad \int^\infty R^{-1}(t)\,\mathrm{d}t = \infty \tag{1.4}$$

or

$$\int_t^\infty R^{-1}(s)\,\mathrm{d}s \int_T^t \tilde{c}(s)h^p(s)\,\mathrm{d}s \quad \text{if} \quad \int^\infty R^{-1}(t)\,\mathrm{d}t < \infty, \tag{1.5}$$

where h is a solution of (1.1) (or a function which is asymptotically close to a solution of (1.1)) and $R = rh^2|h'|^{p-2}$. Criteria of this type can be found in [1–3, 5–7, 9, 10, 13], see also the references therein. Note that the divergence or convergence of the integral $\int^\infty R^{-1}(t)\,\mathrm{d}t$ is closely connected with the so called principality of the solution h of (1.1), see [4, 8] for details.

Let us summarize the known results concerning the above mentioned criteria which apply to perturbations of the Euler and Rieman–Weber type equations. Denote

$$\gamma_p := \left(\frac{p-1}{p}\right)^p, \qquad \mu_p = \frac{1}{2}\left(\frac{p-1}{p}\right)^{p-1}.$$

An example of a nonoscillatory equation of the form (1.1) is the half-linear Euler type equation with the critical coefficient γ_p (called also the oscillation constant)

$$(\Phi(x'))' + \frac{\gamma_p}{t^p}\Phi(x) = 0, \tag{1.6}$$

whose principal solution is $h_1(t) = t^{\frac{p-1}{p}}$ and the second one (linearly independent of h_1) is asymptotically equivalent to $h_2(t) = t^{\frac{p-1}{p}} \log^{\frac{2}{p}} t$, see [11]. Note that the criticality of γ_p in (1.6) means that if we replace γ_p in (1.6) by another constant γ, then (1.6) is oscillatory for $\gamma > \gamma_p$ and nonoscillatory for $\gamma < \gamma_p$. It was shown in [7] that the perturbed Euler type equation

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \tilde{c}(t)\right)\Phi(x) = 0 \tag{1.7}$$

is nonoscillatory if

$$\limsup_{t\to\infty} E(t) < \mu_p, \qquad \liminf_{t\to\infty} E(t) > -3\mu_p$$

and oscillatory if

$$\liminf_{t\to\infty} E(t) > \mu_p,$$

where $E(t) = \log t \int_t^\infty \tilde{c}(s)s^{p-1}\,\mathrm{d}s$. Došlý and Řezníčková [9] proved the same couple of nonoscillation and oscillation criteria with $E(t) = \frac{1}{\log t}\int_T^t \tilde{c}(s)s^{p-1}\log^2 s\,\mathrm{d}s$. Compare both cases of $E(t)$ with (1.4) and (1.5) taking $h(t) = h_1(t)$ and $h(t) = h_2(t)$, respectively.

Further natural step was to find similar statements also for perturbations of the Riemann–Weber (sometimes called Euler–Weber) half-linear equation with critical coefficients

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \frac{\mu_p}{t^p \log^2 t}\right)\Phi(x) = 0. \tag{1.8}$$

This equation has a pair of solutions asymptotically close to the functions $h_1(t) = t^{\frac{p-1}{p}}\log^{\frac{1}{p}} t$ and $h_2(t) = t^{\frac{p-1}{p}}\log^{\frac{1}{p}} t \log^{\frac{2}{p}}(\log t)$ and if we replace the constant μ_p in (1.8) by a different constant μ, then (1.8) is oscillatory for $\mu > \mu_p$ and nonoscillatory for $\mu < \mu_p$, see [12]. The (non)oscillation criteria for the perturbed equation

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \frac{\mu_p}{t^p \log^2 t} + \tilde{c}(t)\right)\Phi(x) = 0 \tag{1.9}$$

were formulated in terms of

$$E(t) = \log(\log t)\int_t^\infty \tilde{c}(s)s^{p-1}\log s\,\mathrm{d}s,$$

which complies with (1.4) taking $h(t) = h_1(t)$. The relevant nonoscillation criterion for (1.9) was proved in [2] and oscillatory criterion in [10]. The case which corresponds to (1.5) and to the second function h_2 remained open.

Recently, the criteria from [2, 10] were generalized in [3] to perturbations of the following generalized Riemann–Weber half-linear equation with critical coefficients

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \sum_{j=1}^n \frac{\mu_p}{t^p \mathrm{Log}_j^2 t}\right)\Phi(x) = 0, \tag{1.10}$$

where $n \in \mathbb{N}$ and

$$\log_1 t = \log t, \qquad \log_k t = \log_{k-1}(\log t), \quad k \geq 2, \qquad \mathrm{Log}_j t = \Pi_{k=1}^j \log_k t.$$

Elbert and Schneider in [12] derived the asymptotic formulas for the two linearly independent nonoscillatory solutions of (1.10). These solutions are asymptotically equivalent to the functions

$$h_1(t) = t^{\frac{p-1}{p}}\mathrm{Log}_n^{\frac{1}{p}} t, \qquad h_2(t) = t^{\frac{p-1}{p}}\mathrm{Log}_n^{\frac{1}{p}} t \log_{n+1}^{\frac{2}{p}} t. \tag{1.11}$$

Došlý in [3] studied the equation

$$L_{RW}[x] := (\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \sum_{j=1}^n \frac{\mu_p}{t^p \mathrm{Log}_j^2 t} + \tilde{c}(t)\right)\Phi(x) = 0 \tag{1.12}$$

and proved the following statement.

Theorem A. *Suppose that the integral* $\int^{\infty} \tilde{c}(t)t^{p-1} \operatorname{Log}_n t \, dt$ *is convergent.*

(i) *If*

$$\limsup_{t\to\infty} \log_{n+1} t \int_t^{\infty} \tilde{c}(s)s^{p-1} \operatorname{Log}_n s \, ds < \mu_p,$$
$$\liminf_{t\to\infty} \log_{n+1} t \int_t^{\infty} \tilde{c}(s)s^{p-1} \operatorname{Log}_n s \, ds > -3\mu_p,$$

then (1.12) *is nonoscillatory.*

(ii) *Suppose that there exists a constant* $\gamma > \frac{2\gamma_p p(p-2)}{3(p-1)^2}$ *such that* $\tilde{c}(t)t^p \log^3 t \geq \gamma$ *for large* t *and*

$$\liminf_{t\to\infty} \log_{n+1} t \int_t^{\infty} \tilde{c}(s)s^{p-1} \operatorname{Log}_n s \, ds > \mu_p.$$

Then (1.12) *is oscillatory.*

Observe that the integral expression from Theorem A relates to (1.4) with $h(t) = h_1(t)$ from (1.11). If $n = 1$, then (1.12) reduces to (1.9) and the criteria from Theorem A reduce to that obtained in [2, 10].

The aim of this paper is to complement Theorem A (and also the corresponding results of [2, 10] in case $n = 1$). We utilize the second function h_2 from (1.11) and find a related couple of criteria for equation (1.12) formulated in terms of the expression

$$\frac{1}{\log_{n+1} t} \int^t \tilde{c}(s)s^{p-1} \operatorname{Log}_n s \log_{n+1}^2 s \, ds$$

which corresponds to (1.5).

2 Auxiliary statements

In this section we present the known statements which will be used in the proofs of our main results in the next section. Denote

$$R(t) := r(t)h^2(t)|h'(t)|^{p-2}, \qquad G(t) := r(t)h(t)\Phi(h'(t)) \tag{2.1}$$

and recall that $q = \frac{p}{p-1}$ is the so called conjugate number of p.

The following statement comes from [13].

Theorem B. *Let* h *be a function such that* $h(t) > 0$ *and* $h'(t) \neq 0$*, both for large* t*. Suppose that the following conditions hold:*

$$\int^{\infty} R^{-1}(t) \, dt < \infty, \qquad \lim_{t\to\infty} G(t) \int_t^{\infty} R^{-1}(s) \, ds = \infty. \tag{2.2}$$

If

$$\limsup_{t\to\infty} \int_t^{\infty} R^{-1}(s) \, ds \int_T^t h(s)L[h](s) \, ds < \frac{1}{q}(-\alpha + \sqrt{2\alpha}), \tag{2.3}$$
$$\liminf_{t\to\infty} \int_t^{\infty} R^{-1}(s) \, ds \int_T^t h(s)L[h](s) \, ds > \frac{1}{q}(-\alpha - \sqrt{2\alpha}) \tag{2.4}$$

for some $\alpha > 0$*, then* (1.1) *is nonoscillatory.*

The following theorem was proved in [6].

Theorem C. *Let h be a positive continuously differentiable function satisfying the following conditions:*

$$h(t)L(h)(t) \geq 0 \quad \textit{for large } t, \qquad \int^{\infty} h(t)L(h)(t)\,\mathrm{d}t = \infty, \tag{2.5}$$

$$\int^{\infty} R^{-1}(t)\,\mathrm{d}t = \infty \quad \textit{and} \quad \lim_{t\to\infty} G(t) = \infty. \tag{2.6}$$

Then (1.1) is oscillatory.

In the following lemma we summarize some technical facts which are either evident or were derived in [3].

Lemma 2.1. *For $n \geq 2$ and large t we have*

$$\operatorname{Log}_n t > \cdots > \operatorname{Log}_1 t = \log t > \cdots > \log_n t$$

and

$$(\log_n t)' = \frac{1}{t \operatorname{Log}_{n-1} t}, \qquad (\operatorname{Log}_n t)' = \frac{\operatorname{Log}_n t}{t} \sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i t}.$$

Moreover, for $h(t) = t^{\frac{p-1}{p}} \operatorname{Log}_n^{\frac{1}{p}} t$ and the operator defined in (1.12) we have

$$h'(t) = \frac{p-1}{p} t^{-\frac{1}{p}} \operatorname{Log}_n^{\frac{1}{p}} t \left(1 + \sum_{i=1}^{n} \frac{1}{(p-1)\operatorname{Log}_i t}\right)$$

and

$$h(t)L_{RW}[h](t) = \frac{\operatorname{Log}_n t}{t \log^3 t}\left[\frac{2\gamma_p p(2-p)}{3(p-1)^2} + \tilde{c}(t)t^p \log^3 t + o(1)\right] \quad \textit{as } t \to \infty. \tag{2.7}$$

3 Main results

Our main result concerning nonoscillation of (1.12) reads as follows.

Theorem 3.1. *If*

$$\limsup_{t\to\infty} \frac{1}{\log_{n+1} t} \int_T^t \tilde{c}(s)s^{p-1} \operatorname{Log}_n s \log_{n+1}^2 s\,\mathrm{d}s < 2\mu_p(-\alpha + \sqrt{2\alpha}), \tag{3.1}$$

$$\liminf_{t\to\infty} \frac{1}{\log_{n+1} t} \int_T^t \tilde{c}(s)s^{p-1} \operatorname{Log}_n s \log_{n+1}^2 s\,\mathrm{d}s > 2\mu_p(-\alpha - \sqrt{2\alpha}) \tag{3.2}$$

for some $\alpha > 0$, then equation (1.12) is nonoscillatory.

Proof. We prove the statement with the use of the function $h(t) = t^{\frac{p-1}{p}} \operatorname{Log}_n^{\frac{1}{p}} t \log_{n+1}^{\frac{2}{p}} t$ in Theorem B. By a direct differentiation (and using Lemma 2.1) we have

$$\begin{aligned}
h'(t) &= \frac{p-1}{p} t^{-\frac{1}{p}} \operatorname{Log}_n^{\frac{1}{p}} t \log_{n+1}^{\frac{2}{p}} t + \frac{1}{p} t^{\frac{p-1}{p}} \operatorname{Log}_n^{\frac{1}{p}-1} t \frac{\operatorname{Log}_n t}{t}\left(\frac{1}{\log t} + \cdots + \frac{1}{\operatorname{Log}_n t}\right)\log_{n+1}^{\frac{2}{p}} t \\
&\quad + \frac{2}{p} t^{\frac{p-1}{p}} \operatorname{Log}_n^{\frac{1}{p}} t \log_{n+1}^{\frac{2}{p}-1} t \frac{1}{t \operatorname{Log}_n t} \\
&= \frac{p-1}{p} t^{-\frac{1}{p}} \operatorname{Log}_n^{\frac{1}{p}} t \log_{n+1}^{\frac{2}{p}} t \left(1 + \sum_{i=1}^{n} \frac{1}{(p-1)\operatorname{Log}_i t} + \frac{2}{(p-1)\operatorname{Log}_{n+1} t}\right).
\end{aligned}$$

Denote $\Gamma_p = \left(\frac{p-1}{p}\right)^{p-1}$. Then

$$\Phi(h') = \Gamma_p t^{-1+\frac{1}{p}} \operatorname{Log}_n^{1-\frac{1}{p}} t \log_{n+1}^{2-\frac{2}{p}} t \left(1 + \sum_{i=1}^{n} \frac{1}{(p-1)\operatorname{Log}_i t} + \frac{2}{(p-1)\operatorname{Log}_{n+1} t}\right)^{p-1}.$$

By a direct differentiation (and using Lemma 2.1 again) we obtain

$$\begin{aligned}
(\Phi(h'))' = &-\gamma_p t^{-2+\frac{1}{p}} \operatorname{Log}_n^{1-\frac{1}{p}} t \log_{n+1}^{2-\frac{2}{p}} t \left(1 + \sum_{i=1}^{n} \frac{1}{(p-1)\operatorname{Log}_i t} + \frac{2}{(p-1)\operatorname{Log}_{n+1} t}\right)^{p-1} \\
&+ \gamma_p t^{-2+\frac{1}{p}} \operatorname{Log}_n^{1-\frac{1}{p}} t \log_{n+1}^{2-\frac{2}{p}} t \sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i t} \left(1 + \sum_{i=1}^{n} \frac{1}{(p-1)\operatorname{Log}_i t} + \frac{2}{(p-1)\operatorname{Log}_{n+1} t}\right)^{p-1} \\
&+ 2\gamma_p t^{-2+\frac{1}{p}} \operatorname{Log}_n^{-\frac{1}{p}} t \log_{n+1}^{1-\frac{2}{p}} t \left(1 + \sum_{i=1}^{n} \frac{1}{(p-1)\operatorname{Log}_i t} + \frac{2}{(p-1)\operatorname{Log}_{n+1} t}\right)^{p-1} \\
&+ \Gamma_p (p-1) t^{-1+\frac{1}{p}} \operatorname{Log}_n^{1-\frac{1}{p}} t \log_{n+1}^{2-\frac{2}{p}} t \left(1 + \sum_{i=1}^{n} \frac{1}{(p-1)\operatorname{Log}_i t} + \frac{2}{(p-1)\operatorname{Log}_{n+1} t}\right)^{p-2} \\
&\times \frac{-1}{(p-1)t} \left[\frac{1}{\log^2 t} + \frac{1}{\operatorname{Log}_2 t}\left(\frac{1}{\log t} + \frac{1}{\operatorname{Log}_2 t}\right) + \cdots + \frac{1}{\operatorname{Log}_n t} \sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i t} \right. \\
&\qquad \left. + \frac{2}{\operatorname{Log}_{n+1} t} \sum_{i=1}^{n+1} \frac{1}{\operatorname{Log}_i t}\right].
\end{aligned}$$

Observe that the expression in the square brackets can be rearranged as follows:

$$\sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i^2 t} + \sum_{1 \le i < j \le n} \frac{1}{\operatorname{Log}_i t \operatorname{Log}_j t} + \frac{2}{\operatorname{Log}_{n+1} t} \sum_{i=1}^{n+1} \frac{1}{\operatorname{Log}_i t}.$$

Hence

$$\begin{aligned}
(\Phi(h'))' = t^{-2+\frac{1}{p}} &\operatorname{Log}_n^{1-\frac{1}{p}} t \log_{n+1}^{2-\frac{2}{p}} t \left(1 + \sum_{i=1}^{n} \frac{1}{(p-1)\operatorname{Log}_i t} + \frac{2}{(p-1)\operatorname{Log}_{n+1} t}\right)^{p-2} \\
\times \Bigg\{ &-\gamma_p \left(1 + \sum_{i=1}^{n} \frac{1}{(p-1)\operatorname{Log}_i t} + \frac{2}{(p-1)\operatorname{Log}_{n+1} t}\right) \\
&+ \gamma_p \sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i t} \left(1 + \sum_{i=1}^{n} \frac{1}{(p-1)\operatorname{Log}_i t} + \frac{2}{(p-1)\operatorname{Log}_{n+1} t}\right) \\
&+ 2\gamma_p \frac{1}{\operatorname{Log}_{n+1} t} \left(1 + \sum_{i=1}^{n} \frac{1}{(p-1)\operatorname{Log}_i t} + \frac{2}{(p-1)\operatorname{Log}_{n+1} t}\right) \\
&-\Gamma_p \left[\sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i^2 t} + \sum_{1 \le i < j \le n} \frac{1}{\operatorname{Log}_i t \operatorname{Log}_j t} + \frac{2}{\operatorname{Log}_{n+1} t} \sum_{i=1}^{n+1} \frac{1}{\operatorname{Log}_i t}\right] \Bigg\}.
\end{aligned}$$

Denote by $A(t)$ the expression in the curly brackets. By a direct computation with using the fact that

$$\left(\sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i t}\right)^2 = \sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i^2 t} + \sum_{1 \le i < j \le n} \frac{2}{\operatorname{Log}_i t \operatorname{Log}_j t}$$

we obtain

$$\begin{aligned} A(t) = & -\gamma_p + \gamma_p \frac{p-2}{p-1} \sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i t} + 2\gamma_p \frac{p-2}{p-1} \frac{1}{\operatorname{Log}_{n+1} t} - \gamma_p \sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i^2 t} \\ & + \gamma_p \frac{2-p}{p-1} \sum_{1 \le i < j \le n} \frac{1}{\operatorname{Log}_i t \operatorname{Log}_j t} + 2\gamma_p \frac{2-p}{p-1} \frac{1}{\operatorname{Log}_{n+1} t} \sum_{i=1}^{n+1} \frac{1}{\operatorname{Log}_i t}. \end{aligned} \tag{3.3}$$

Next, denote

$$B(t) := \left(1 + \sum_{i=1}^{n} \frac{1}{(p-1)\operatorname{Log}_i t} + \frac{2}{(p-1)\operatorname{Log}_{n+1} t}\right)^{p-2}.$$

Using the power expansion

$$(1+x)^s = 1 + sx + \frac{s(s-1)}{2}x^2 + \frac{s(s-1)(s-2)}{6}x^3 + o(x^3) \quad \text{as } x \to 0,\ s \in \mathbb{R},$$

we obtain

$$\begin{aligned} B(t) = 1 + \frac{p-2}{p-1}\left(\sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i t} + \frac{2}{\operatorname{Log}_{n+1} t}\right) + \frac{(p-2)(p-3)}{2(p-1)^2}\left(\sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i t} + \frac{2}{\operatorname{Log}_{n+1} t}\right)^2 \\ + \frac{(p-2)(p-3)(p-4)}{6(p-1)^3}\left(\sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i t} + \frac{2}{\operatorname{Log}_{n+1} t}\right)^3 + o(\log^{-3} t), \end{aligned}$$

as $t \to \infty$.

Next observe that if at least one of the indices i, j, k is greater than one, then

$$\frac{1}{\operatorname{Log}_i t \operatorname{Log}_j t \operatorname{Log}_k t} = o\left(\log^{-3} t\right) \quad \text{as } t \to \infty.$$

Hence we can write $B(t)$ in the form

$$\begin{aligned} B(t) = 1 & + \frac{p-2}{p-1}\left(\sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i t} + \frac{2}{\operatorname{Log}_{n+1} t}\right) \\ & + \frac{(p-2)(p-3)}{2(p-1)^2}\left(\sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i^2 t} + \sum_{1 \le i < j \le n} \frac{2}{\operatorname{Log}_i t \operatorname{Log}_j t} + \frac{4}{\operatorname{Log}_{n+1} t}\sum_{i=1}^{n+1} \frac{1}{\operatorname{Log}_i t}\right) \\ & + \frac{(p-2)(p-3)(p-4)}{6(p-1)^3} \frac{1}{\log^3 t} + o(\log^{-3} t) \end{aligned} \tag{3.4}$$

as $t \to \infty$.

From (3.3) and (3.4), we obtain

$$\begin{aligned} & A(t) \cdot B(t) \\ & \quad = -\gamma_p + \gamma_p \frac{p-2}{p-1} \sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i t} + 2\gamma_p \frac{p-2}{p-1} \frac{1}{\operatorname{Log}_{n+1} t} - \gamma_p \sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i^2 t} \\ & \qquad + \gamma_p \frac{2-p}{p-1} \sum_{1 \le i < j \le n} \frac{1}{\operatorname{Log}_i t \operatorname{Log}_j t} + 2\gamma_p \frac{2-p}{p-1} \frac{1}{\operatorname{Log}_{n+1} t} \sum_{i=1}^{n+1} \frac{1}{\operatorname{Log}_i t} \\ & \qquad - \gamma_p \frac{p-2}{p-1} \sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i t} + \gamma_p \left(\frac{p-2}{p-1}\right)^2 \sum_{i=1}^{n} \frac{1}{\operatorname{Log}_i^2 t} + 2\gamma_p \left(\frac{p-2}{p-1}\right)^2 \sum_{1 \le i < j \le n} \frac{1}{\operatorname{Log}_i t \operatorname{Log}_j t} \end{aligned}$$

$$
\begin{aligned}
&+2\gamma_p\left(\frac{p-2}{p-1}\right)^2\frac{1}{\operatorname{Log}_{n+1}t}\sum_{i=1}^{n}\frac{1}{\operatorname{Log}_i t}-\gamma_p\frac{p-2}{p-1}\frac{1}{\log^3 t}\\
&-2\gamma_p\frac{p-2}{p-1}\frac{1}{\operatorname{Log}_{n+1}t}+2\gamma_p\left(\frac{p-2}{p-1}\right)^2\frac{1}{\operatorname{Log}_{n+1}t}\sum_{i=1}^{n}\frac{1}{\operatorname{Log}_i t}+4\gamma_p\left(\frac{p-2}{p-1}\right)^2\frac{1}{\operatorname{Log}^2_{n+1}t}\\
&-\gamma_p\frac{(p-2)(p-3)}{2(p-1)^2}\sum_{i=1}^{n}\frac{1}{\operatorname{Log}^2_i t}+\gamma_p\frac{(p-2)^2(p-3)}{2(p-1)^3}\frac{1}{\log^3 t}\\
&-\gamma_p\frac{(p-2)(p-3)}{(p-1)^2}\sum_{1\le i<j\le n}\frac{1}{\operatorname{Log}_i t\operatorname{Log}_j t}-2\gamma_p\frac{(p-2)(p-3)}{(p-1)^2}\frac{1}{\operatorname{Log}_{n+1}t}\sum_{i=1}^{n+1}\frac{1}{\operatorname{Log}_i t}\\
&-\gamma_p\frac{(p-2)(p-3)(p-4)}{6(p-1)^3}\frac{1}{\log^3 t}+o(\log^{-3}t)\\
=&-\gamma_p-\mu_p\sum_{i=1}^{n}\frac{1}{\operatorname{Log}^2_i t}-\frac{2\gamma_p p(p-2)}{3(p-1)^2}\frac{1}{\log^3 t}+o(\log^{-3}t)
\end{aligned}
$$

as $t\to\infty$. Summarizing the above computations, we have

$$
(\Phi(h'))'=t^{-2+\frac{1}{p}}\operatorname{Log}_n^{1-\frac{1}{p}}t\log_{n+1}^{2-\frac{2}{p}}t\left(-\gamma_p-\mu_p\sum_{i=1}^{n}\frac{1}{\operatorname{Log}^2_i t}-\frac{2\gamma_p p(p-2)}{3(p-1)^2}\frac{1}{\log^3 t}+o(\log^{-3}t)\right)
$$

as $t\to\infty$. Consequently, for the operator L_{RW} defined in (1.12) we have

$$
\begin{aligned}
hL_{RW}[h]&=h(\Phi(h'))'+h^p\left(\frac{\gamma_p}{t^p}+\sum_{j=1}^{n}\frac{\mu_p}{t^p\operatorname{Log}^2_j t}+\tilde c(t)\right)\\
&=\frac{\operatorname{Log}_n t\log^2_{n+1}t}{t}\left(-\gamma_p-\mu_p\sum_{i=1}^{n}\frac{1}{\operatorname{Log}^2_i t}-\frac{2\gamma_p p(p-2)}{3(p-1)^2}\frac{1}{\log^3 t}+o(\log^{-3}t)\right)\\
&\quad+t^{p-1}\operatorname{Log}_n t\log^2_{n+1}t\left(\frac{\gamma_p}{t^p}+\sum_{j=1}^{n}\frac{\mu_p}{t^p\operatorname{Log}^2_j t}+\tilde c(t)\right)\\
&=\frac{\operatorname{Log}_n t\log^2_{n+1}t}{t\log^3 t}\left(-\frac{2\gamma_p p(p-2)}{3(p-1)^2}+o(1)\right)+\tilde c(t)t^{p-1}\operatorname{Log}_n t\log^2_{n+1}t
\end{aligned}
\tag{3.5}
$$

as $t\to\infty$. In order to check conditions (2.2), express $R(t)$ and $G(t)$ from (2.1):

$$
\begin{aligned}
R(t)&=h^2(t)|h'(t)|^{p-2}\\
&=\left(\frac{p-1}{p}\right)^{p-2}t\operatorname{Log}_n t\log^2_{n+1}t\left(1+\sum_{i=1}^{n}\frac{1}{(p-1)\operatorname{Log}_i t}+\frac{2}{(p-1)\operatorname{Log}_{n+1}t}\right)^{p-2}\\
&=\left(\frac{p-1}{p}\right)^{p-2}t\operatorname{Log}_n t\log^2_{n+1}t(1+o(1))
\end{aligned}
$$

and

$$
\begin{aligned}
G(t)&=h(t)\Phi(h'(t))\\
&=\left(\frac{p-1}{p}\right)^{p-1}\operatorname{Log}_n t\log^2_{n+1}t\left(1+\sum_{i=1}^{n}\frac{1}{(p-1)\operatorname{Log}_i t}+\frac{2}{(p-1)\operatorname{Log}_{n+1}t}\right)^{p-1}\\
&=\left(\frac{p-1}{p}\right)^{p-1}\operatorname{Log}_n t\log^2_{n+1}t(1+o(1))
\end{aligned}
$$

as $t \to \infty$. Since $\int^\infty R^{-1}(s)\,\mathrm{d}s < \infty$, the first condition in (2.2) is satisfied. The second condition in (2.2) is also fulfilled, since

$$\int_t^\infty \frac{1}{R(s)}\,\mathrm{d}s = \left(\frac{p}{p-1}\right)^{p-2} \frac{1}{\log_{n+1} t}(1+o(1)) \tag{3.6}$$

and hence

$$G(t)\int_t^\infty \frac{1}{R(s)}\,\mathrm{d}s = \frac{p-1}{p}\operatorname{Log}_n t \log_{n+1} t(1+o(1)) \to \infty$$

as $t \to \infty$.

Finally, we show that conditions (2.3) and (2.4) hold. To this end, let $\varepsilon \in (0,1)$. Then

$$\lim_{t\to\infty} \frac{\operatorname{Log}_n t \log_{n+1}^2 t}{\log^{1+\varepsilon} t} < \lim_{t\to\infty} \frac{\log_2^{n+1} t}{\log^{\varepsilon} t} = \lim_{t\to\infty} \frac{(n+1)!}{\varepsilon^{n+1}\log^{\varepsilon} t} = 0$$

and hence

$$\lim_{t\to\infty} \frac{1}{\log_{n+1} t}\int_T^t \frac{\operatorname{Log}_n s \log_{n+1}^2 s}{s\log^3 s}\,\mathrm{d}s < \lim_{t\to\infty} \frac{1}{\log_{n+1} t}\int_T^t \frac{1}{s\log^{2-\varepsilon} s}\,\mathrm{d}s = 0. \tag{3.7}$$

From (3.5) and (3.6) we obtain

$$\begin{aligned}
&\left(\int_t^\infty R^{-1}(s)\,\mathrm{d}s\right)\left(\int_T^t h(s)L_{RW}[h](s)\,\mathrm{d}s\right)\\
&\quad= \left(\frac{p}{p-1}\right)^{p-2} \frac{1}{\log_{n+1} t}(1+o(1))\\
&\qquad\times \int_T^t \frac{\operatorname{Log}_n s \log_{n+1}^2 s}{s\log^3 s}\left(-\frac{2\gamma_p p(p-2)}{3(p-1)^2}+o(1)\right) + \tilde{c}(s)s^{p-1}\operatorname{Log}_n s \log_{n+1}^2 s\,\mathrm{d}s
\end{aligned}$$

as $t \to \infty$. Conditions (3.1) and (3.2) together with (3.7) imply

$$\begin{aligned}
&\limsup_{t\to\infty}\left(\int_t^\infty R^{-1}(s)\,\mathrm{d}s\right)\left(\int_T^t h(s)L_{RW}[h](s)\,\mathrm{d}s\right)\\
&\quad= \left(\frac{p}{p-1}\right)^{p-2}\limsup_{t\to\infty}\frac{1}{\log_{n+1} t}\int_T^t \tilde{c}(s)s^{p-1}\operatorname{Log}_n s \log_{n+1}^2 s\,\mathrm{d}s < \frac{1}{q}(-\alpha+\sqrt{2\alpha})
\end{aligned}$$

and

$$\begin{aligned}
&\liminf_{t\to\infty}\left(\int_t^\infty R^{-1}(s)\,\mathrm{d}s\right)\left(\int_T^t h(s)L_{RW}[h](s)\,\mathrm{d}s\right)\\
&\quad= \left(\frac{p}{p-1}\right)^{p-2}\liminf_{t\to\infty}\frac{1}{\log_{n+1} t}\int_T^t \tilde{c}(s)s^{p-1}\operatorname{Log}_n s \log_{n+1}^2 s\,\mathrm{d}s > \frac{1}{q}(-\alpha-\sqrt{2\alpha}).
\end{aligned}$$

All assumptions of Theorem B are true, which finishes the proof. □

To obtain the oscillatory counterpart of Theorem 3.1, we first prove the following criterion for the equation

$$\tilde{L}_{RW}[x] := (\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \sum_{j=1}^{n+1}\frac{\mu_p}{t^p\operatorname{Log}_j^2 t} + d(t)\right)\Phi(x) = 0, \tag{3.8}$$

which is in fact equation (1.12) shifted from n to $n+1$. The reason why we formulate this criterion rather for (3.8) than for (1.12) is only technical.

Theorem 3.2. *Suppose that there exists a constant γ such that*

$$d(t)t^p \log^3 t \geq \gamma > \frac{2\gamma_p p(p-2)}{3(p-1)^2} \tag{3.9}$$

for large t. If

$$\int^\infty d(t)t^{p-1} \operatorname{Log}_{n+1} t \, dt = \infty, \tag{3.10}$$

then equation (3.8) is oscillatory.

Proof. Take $h(t) = t^{\frac{p-1}{p}} \operatorname{Log}_{n+1}^{\frac{1}{p}} t$. According to Lemma 2.1 (with n replaced by $n+1$)

$$h'(t) = \frac{p-1}{p} t^{-\frac{1}{p}} \operatorname{Log}_{n+1}^{\frac{1}{p}} t(1+o(1)) \quad \text{as } t \to \infty.$$

Hence, by (2.1)

$$R(t) = h^2(t)|h'(t)|^{p-2} = \left(\frac{p-1}{p}\right)^{p-2} t \operatorname{Log}_{n+1} t(1+o(1)) \quad \text{as } t \to \infty$$

and consequently

$$\int^t R^{-1}(s)\, ds = \left(\frac{p-1}{p}\right)^{2-p} \log_{n+2} t(1+o(1)) \to \infty \quad \text{as } t \to \infty.$$

Further,

$$G(t) = h(t)\Phi(h'(t)) = \left(\frac{p-1}{p}\right)^{p-1} \operatorname{Log}_{n+1} t(1+o(1)) \to \infty \quad \text{as } t \to \infty.$$

Rewriting (2.7) for the operator from (3.8) we have

$$\begin{aligned} h(t)\tilde{L}_{RW}[h](t) &= \frac{\operatorname{Log}_{n+1} t}{t \log^3 t} \left[\frac{2\gamma_p p(2-p)}{3(p-1)^2} + d(t)t^p \log^3 t + o(1)\right] \\ &= \left[\frac{2\gamma_p p(2-p)}{3(p-1)^2} + o(1)\right] \frac{\operatorname{Log}_{n+1} t}{t \log^3 t} + d(t)t^{p-1} \operatorname{Log}_{n+1} t \end{aligned}$$

as $t \to \infty$. Because the integral $\int^\infty \frac{\operatorname{Log}_{n+1} t}{t \log^3 t}\, dt$ is convergent, condition (3.10) implies

$$\int^\infty h\tilde{L}_{RW}[h](t)\, dt = \infty.$$

Thanks to condition (3.9) we have also $h\tilde{L}_{RW}[h](t) \geq 0$ for large t. This means that equation (3.8) is oscillatory by Theorem C. □

The following statement is the oscillatory criterion which complements Theorem 3.1.

Theorem 3.3. *Suppose that there exists a constant γ such that*

$$t^p \log^3 t \left(\tilde{c}(t) - \frac{\mu_p}{t^p \operatorname{Log}_{n+1}^2 t}\right) \geq \gamma > \frac{2\gamma_p p(p-2)}{3(p-1)^2} \tag{3.11}$$

for large t. If

$$\liminf_{t\to\infty} \frac{1}{\log_{n+1} t} \int_T^t \tilde{c}(s)s^{p-1} \operatorname{Log}_n s \log_{n+1}^2 s \, ds > \mu_p, \tag{3.12}$$

then (1.12) is oscillatory.

Proof. Let us rewrite (1.12) into the form

$$(\Phi(x'))' + \left(\frac{\gamma_p}{t^p} + \sum_{j=1}^{n+1} \frac{\mu_p}{t^p \mathrm{Log}_j^2 t} + \left(\tilde{c}(t) - \frac{\mu_p}{t^p\,\mathrm{Log}_{n+1}^2 t}\right)\right)\Phi(x) = 0,$$

so (1.12) is seen as a perturbation of the generalized Riemann–Weber equation with the critical coefficients and with $n+1$ elements in the sum. We apply Theorem 3.2 with the perturbation term $d(t) = \tilde{c}(t) - \frac{\mu_p}{t^p\,\mathrm{Log}_{n+1}^2 t}$. Then (3.9) is guaranteed by (3.11). With respect to (3.12) there exist $\varepsilon > 0$ and $\tilde{T} > T$ such that

$$\liminf_{t\to\infty} \frac{1}{\log_{n+1} t} \int_T^t \tilde{c}(s)s^{p-1}\,\mathrm{Log}_{n+1} s \log_{n+1} s \,\mathrm{d}s > \mu_p + \varepsilon$$

and also

$$\int_T^t \tilde{c}(s)s^{p-1}\,\mathrm{Log}_{n+1} s \log_{n+1} s \,\mathrm{d}s > (\mu_p + \varepsilon)\log_{n+1} t$$

for $t > \tilde{T}$. Let $b > \tilde{T}$. With the use of integration by parts and the above inequality, we have

$$\begin{aligned}
&\int_T^b \left(\tilde{c}(t) - \frac{\mu_p}{t^p\,\mathrm{Log}_{n+1}^2 t}\right) t^{p-1}\,\mathrm{Log}_{n+1} t \,\mathrm{d}t \\
&\quad= \int_T^b \tilde{c}(t)t^{p-1}\,\mathrm{Log}_{n+1} t\,\mathrm{d}t - \int_T^b \frac{\mu_p}{t\,\mathrm{Log}_{n+1} t}\,\mathrm{d}t \\
&\quad= \int_T^b \frac{1}{\log_{n+1} t}\,\tilde{c}(t)t^{p-1}\,\mathrm{Log}_{n+1} t \log_{n+1} t\,\mathrm{d}t - \mu_p\left[\log_{n+2} t\right]_T^b \\
&\quad= \left[\frac{1}{\log_{n+1} t}\int_T^t \tilde{c}(s)s^{p-1}\,\mathrm{Log}_{n+1} s \log_{n+1} s\,\mathrm{d}s\right]_T^b + \int_T^{\tilde{T}} \frac{\int_T^t \tilde{c}(s)s^{p-1}\,\mathrm{Log}_{n+1} s \log_{n+1} s\,\mathrm{d}s}{t\,\mathrm{Log}_n t \log_{n+1}^2 t}\,\mathrm{d}t \\
&\qquad+ \int_{\tilde{T}}^b \frac{\int_T^t \tilde{c}(s)s^{p-1}\,\mathrm{Log}_{n+1} s \log_{n+1} s\,\mathrm{d}s}{t\,\mathrm{Log}_n \log_{n+1}^2 t}\,\mathrm{d}t - \mu_p\left[\log_{n+2} t\right]_T^b \\
&\quad\geq \frac{1}{\log_{n+1} b}\int_T^b \tilde{c}(t)t^{p-1}\,\mathrm{Log}_{n+1} t \log_{n+1} t\,\mathrm{d}t + K_1 + \int_{\tilde{T}}^b \frac{\mu_p+\varepsilon}{t\,\mathrm{Log}_{n+1} t}\,\mathrm{d}t - \mu_p\left[\log_{n+2} t\right]_T^b \\
&\quad\geq \mu_p + \varepsilon + K_1 + (\mu_p+\varepsilon)\left[\log_{n+2} t\right]_{\tilde{T}}^b - \mu_p\left[\log_{n+2} t\right]_T^b \\
&\quad= \mu_p + \varepsilon + K_1 + \varepsilon\log_{n+2} b - K_2 \to \infty
\end{aligned}$$

as $b \to \infty$, where

$$K_1 = \int_T^{\tilde{T}} \frac{\int_T^t \tilde{c}(s)s^{p-1}\,\mathrm{Log}_{n+1} s \log_{n+1} s\,\mathrm{d}s}{t\,\mathrm{Log}_n t \log_{n+1}^2 t}\,\mathrm{d}t, \qquad K_2 = (\mu_p+\varepsilon)\log_{n+2}\tilde{T} - \mu_p\log_{n+2} T.$$

Hence condition (3.10) is satisfied and (1.12) is oscillatory according to Theorem 3.2. □

Remark 3.4. If $\alpha = \frac{1}{2}$ in Theorem 3.1, then

$$2\mu_p(-\alpha + \sqrt{2\alpha}) = \mu_p, \qquad 2\mu_p(-\alpha - \sqrt{2\alpha}) = -3\mu_p$$

and the constants from (3.1) and (3.2) in Theorem 3.1 reduce to the constants in Theorem A, part (i). The generalization for $\alpha \neq \frac{1}{2}$ is due to Theorem B. Note also that the constants in the nonoscillatory part of Theorem A could be generalized in the same way by utilizing [13, Theorem 3.2] in the proof of Theorem A.

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